On Voevodsky's work

Symposium on Algebraic Geometry at Hiroshima

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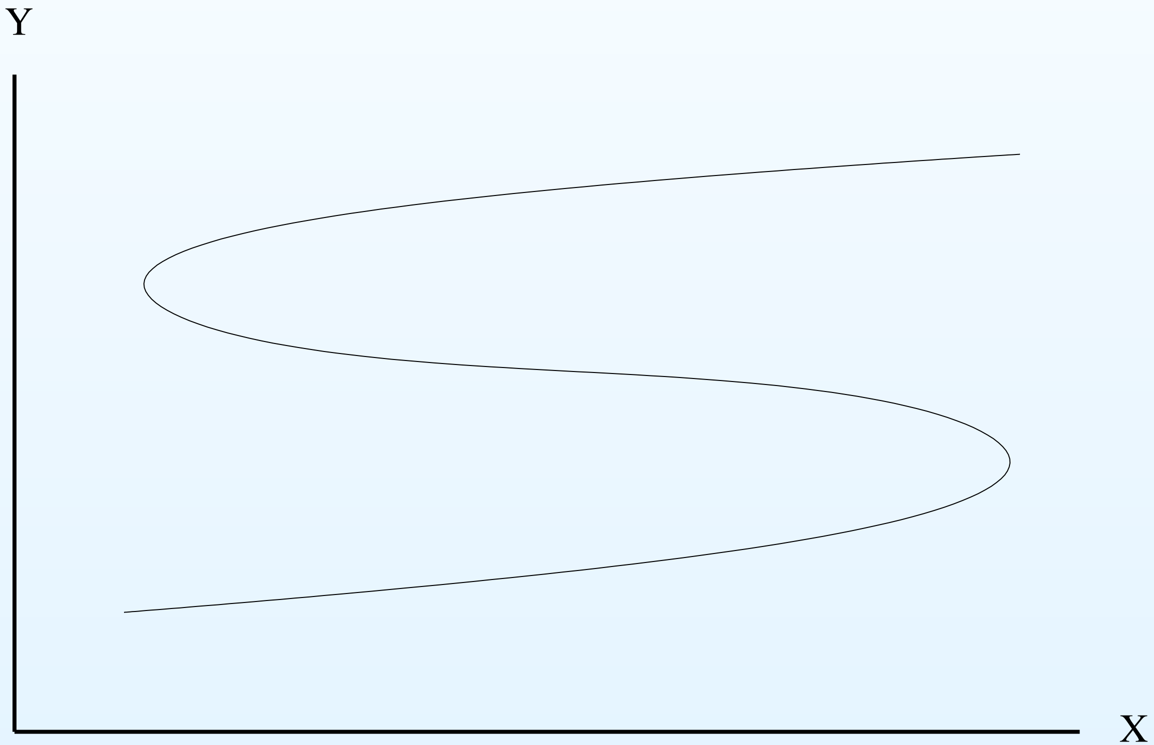
- 2003 Publication of "Motivic cohomology with $\mathbb{Z}/2$-coefficients".

Ingredients and definitions

Correspondences

Let X and Y be two smooth schemes. Then an elementary correspondence from X (connected) to Y is an irreducible closed subset of $X \times Y$ which is finite and surjective over X. A correspondence is an element of the free abelian group $Cor(X, Y)$ generated by the elementary correspondences.



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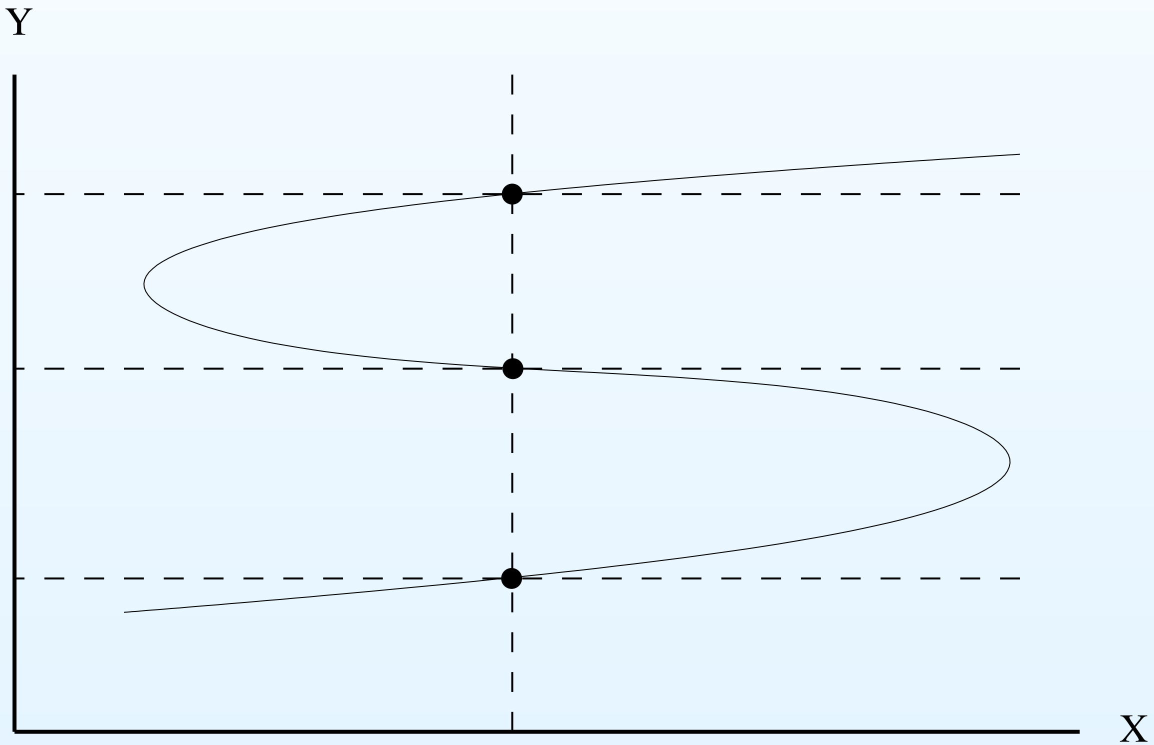

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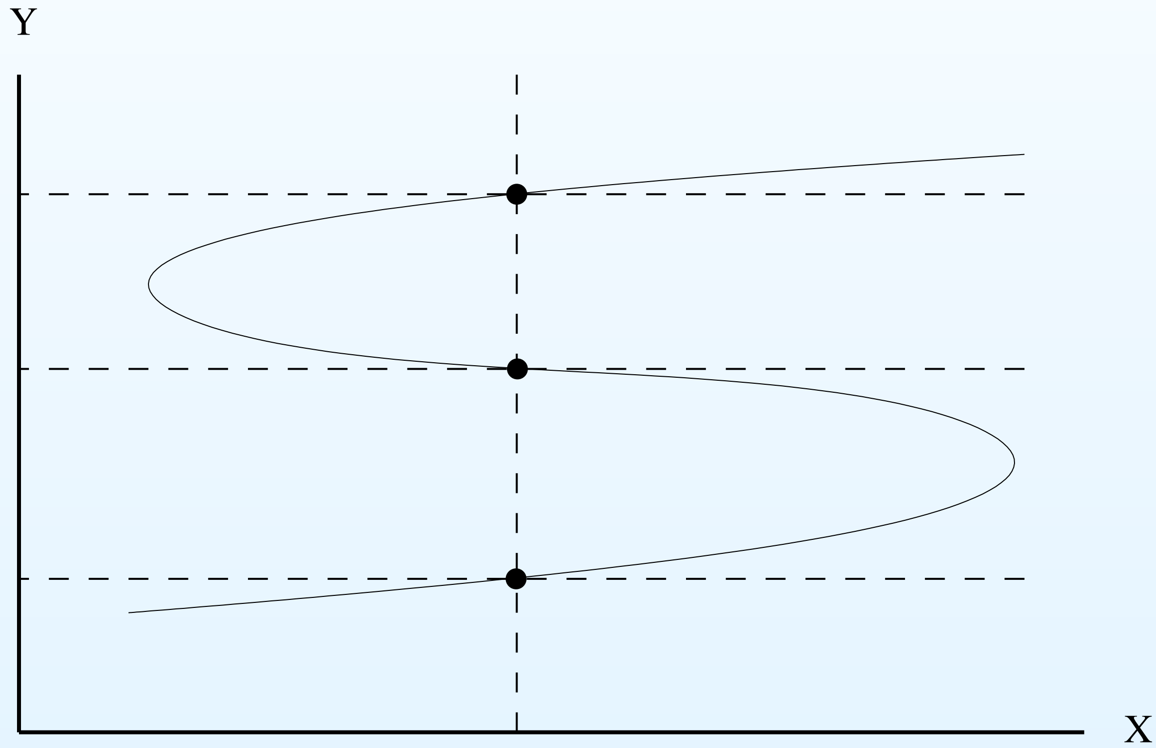
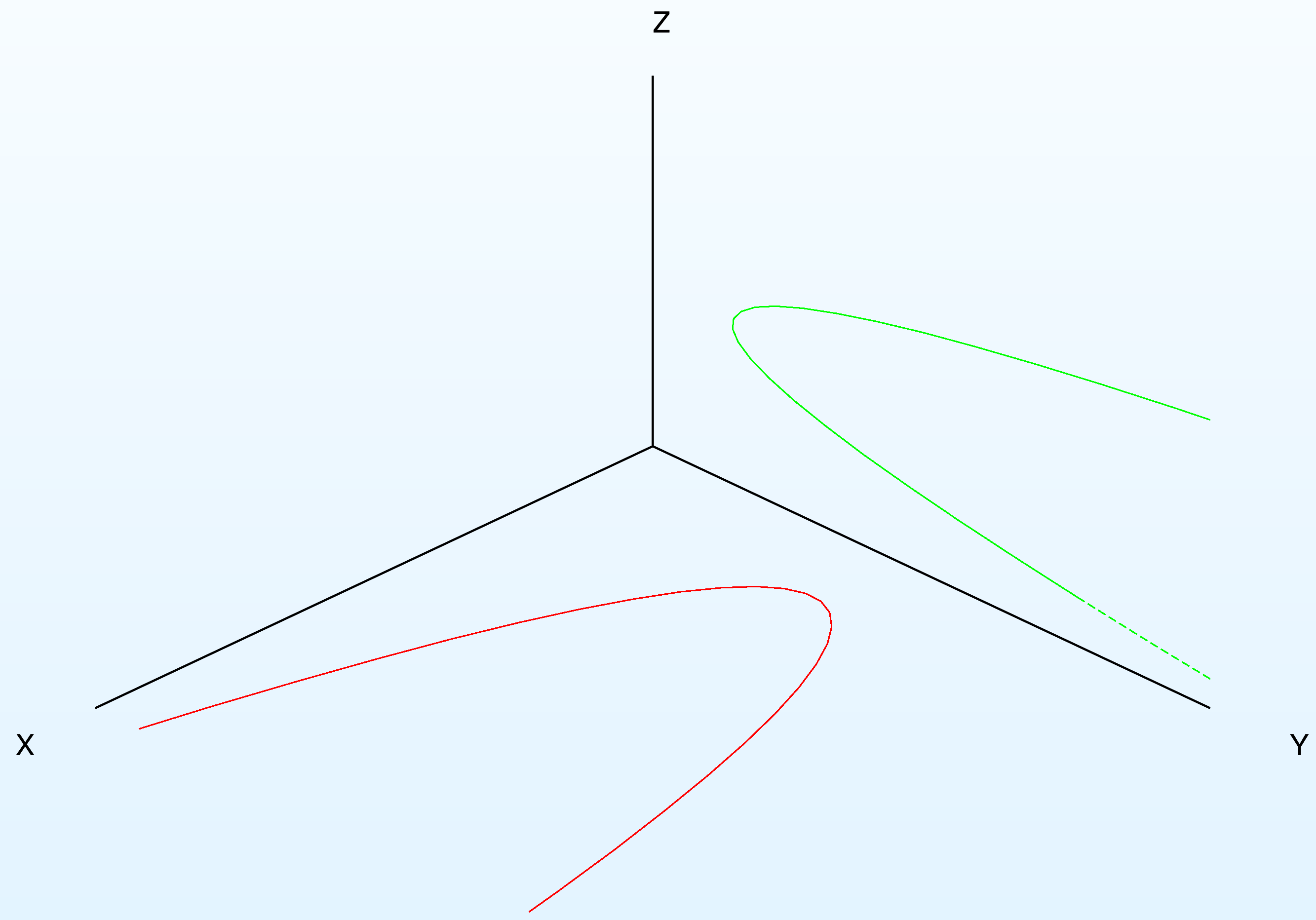


In a formula: $\mathrm{Hom}(S, \coprod_{d=0}^{\infty} S^d(X))^+ \simeq \mathrm{Corr}(S,X)$.

The category $Corr_k$

Let $Corr_k$ be the category of smooth schemes and correspondences as morphism.
What is the composition?



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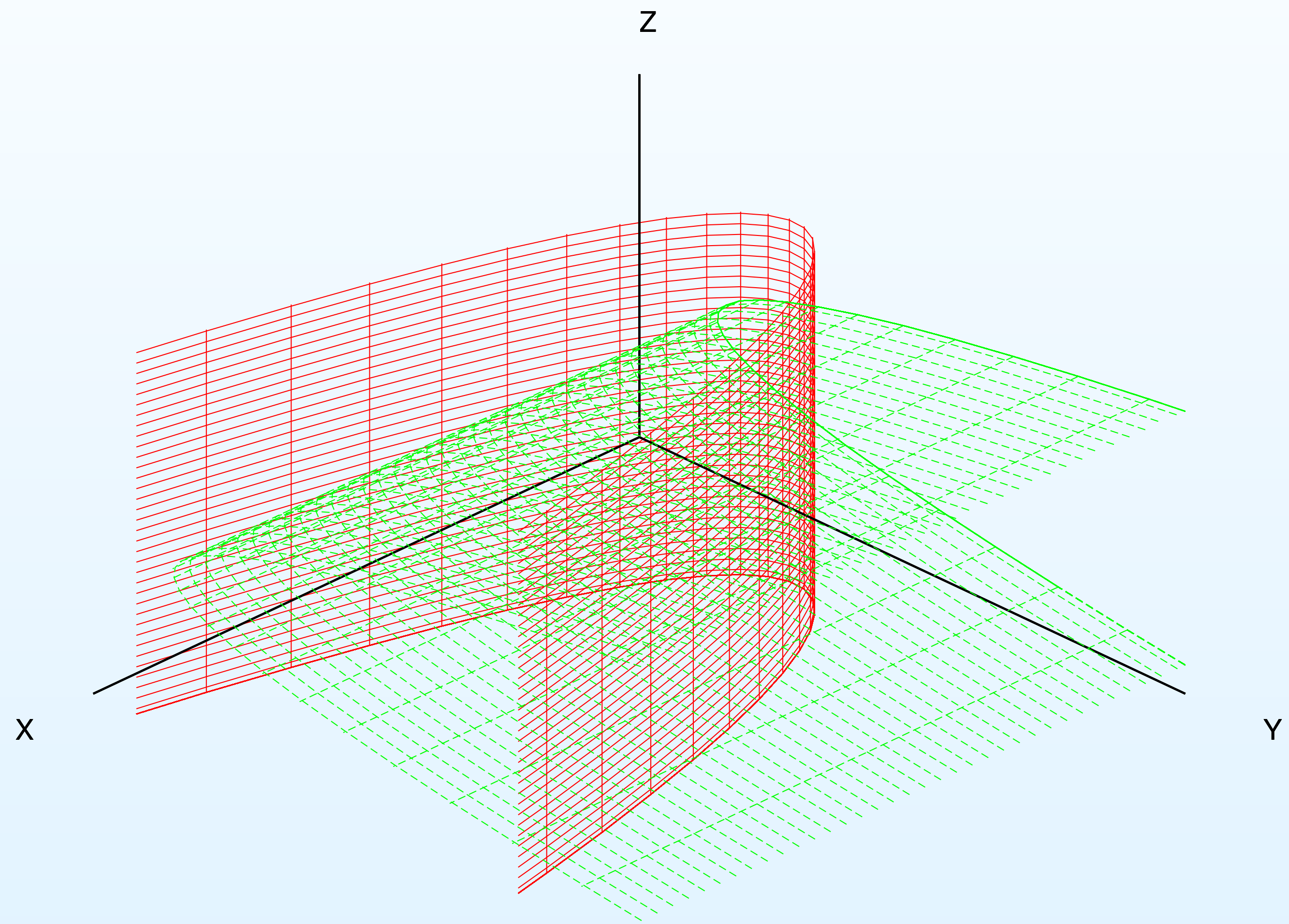

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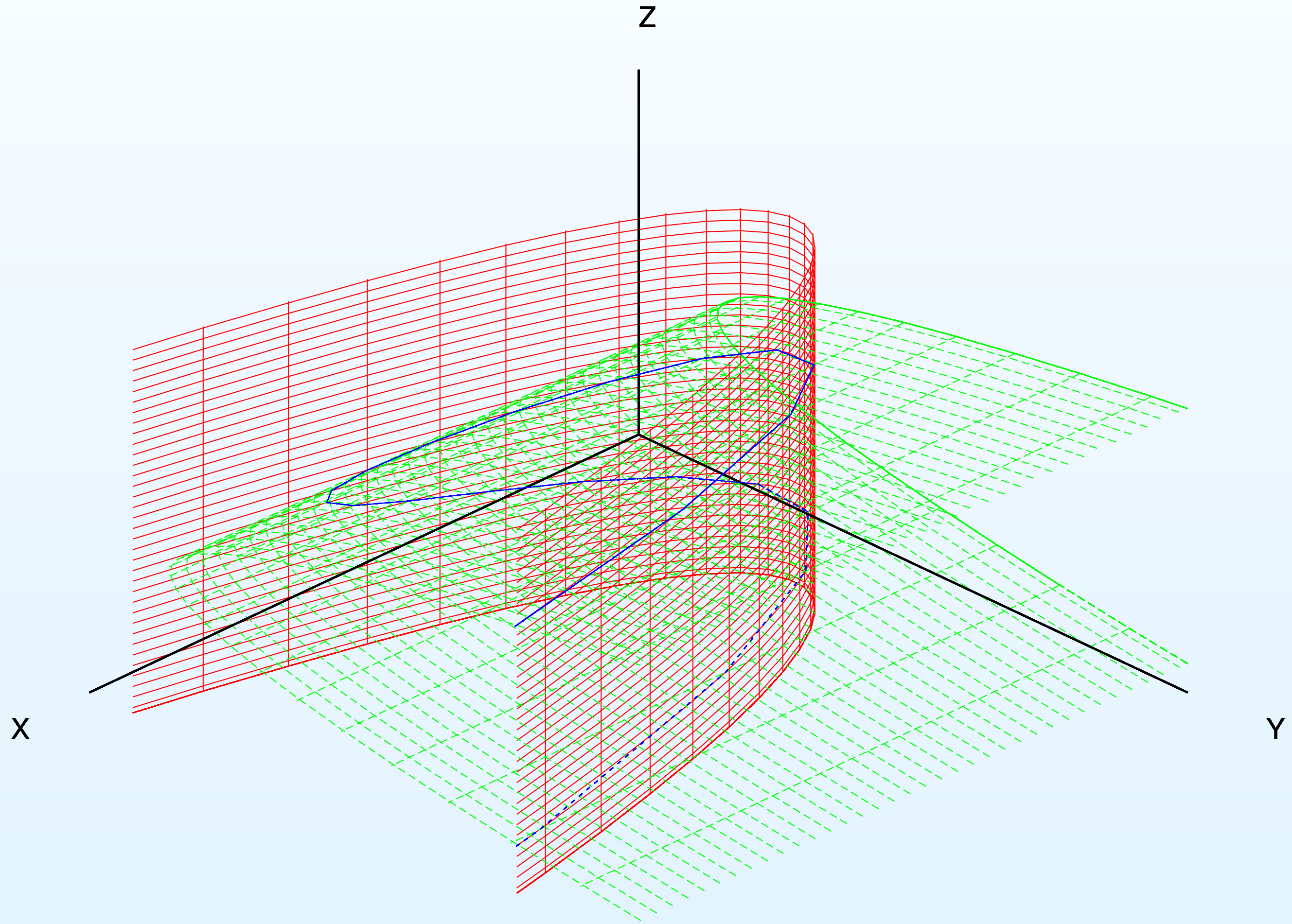

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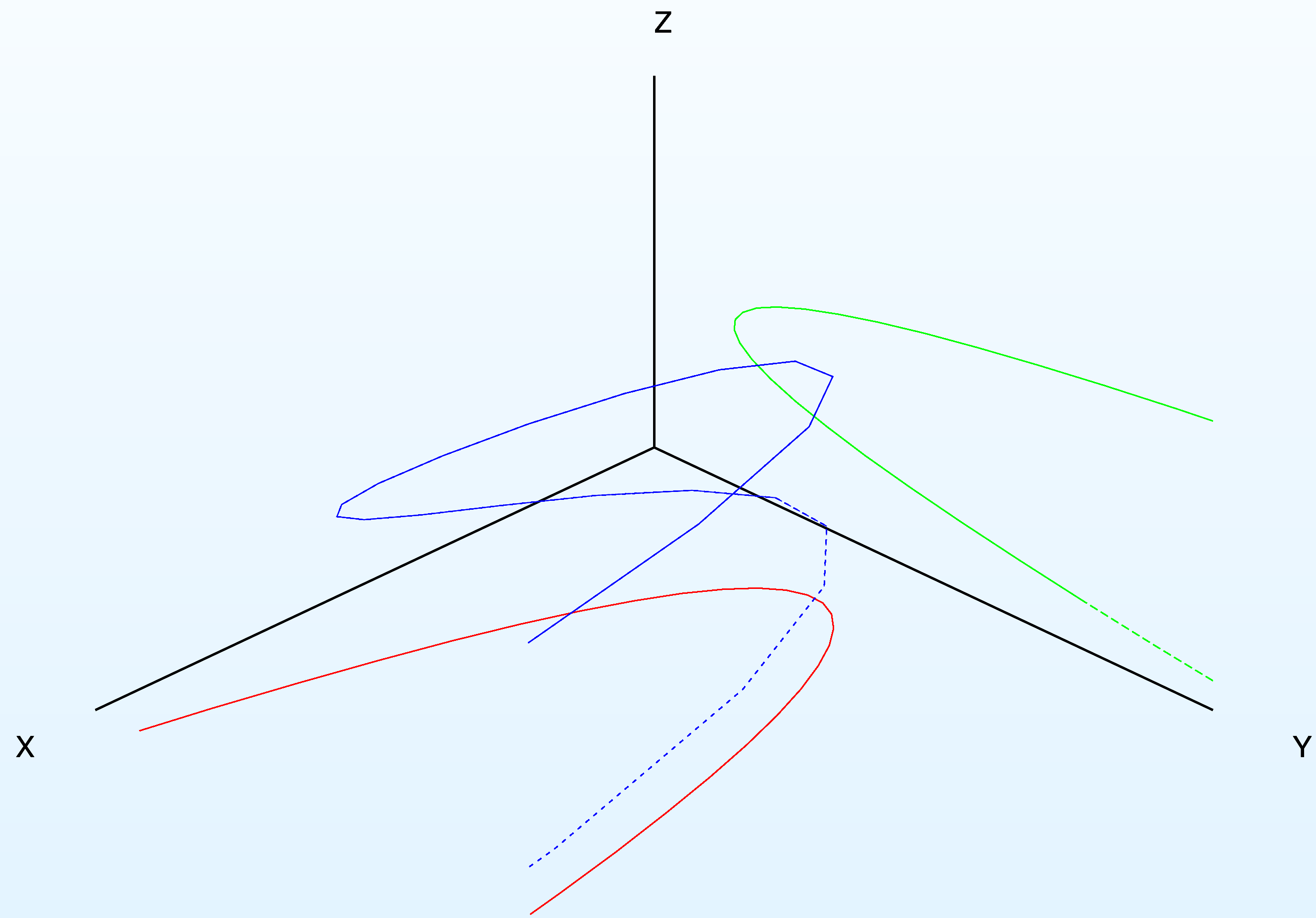

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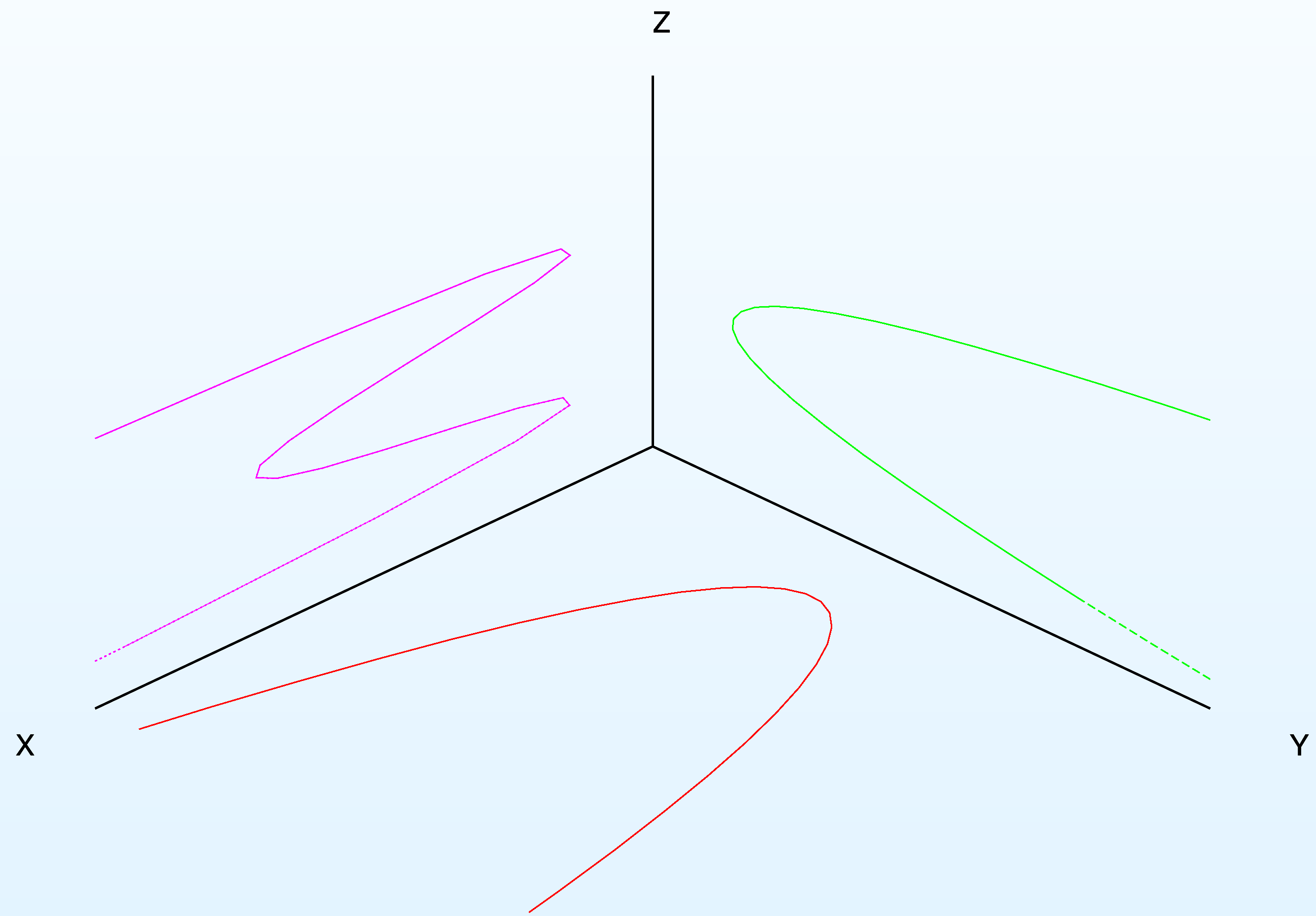

Presheaves with transfers

A presheaf with transfers (PST) is a presheaf on $Corr_k$, i.e., a presheaf on Sm_k such that for every correspondence from X to Y, there is a map $F(Y) \to F(X)$.

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- $\mathcal{O}$, $\mathcal{O}^*$.
- Hecke functors for the absolute Galois group (say, Galois modules)
- (higher) Chow groups (on smooth projective varieties).

Nisnevich topology

The Nisnevich topology is between the Zariski and the étale topology. A map $f : Y \to X$ is a Nisnevich covering if it is an étale covering and for all $x \in X$ there is a $y \in Y$ such that $f(y) = x$ and f induces an isomorphism on the residue fields.

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- Let F be an étale sheaf of $\mathbb{Q}$-modules, then $H^n_{Nis}(-, F) = H^n_{ét}(-, F)$ for all $n \geq 0$. (étale)

The category $\mathrm{DM}^{-}(k)$

The category $\mathbf{DM}^{-}_{eff}(k)$ is the $\mathbb{A}^1$-localization of the derived category of bounded above Nisnevich sheaves with transfers.

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$$\begin{array}{ccccccc}
SmProj_k & & \longrightarrow & & \mathcal{M}(k) & & \\
\downarrow & & & & \downarrow & & \\
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In $\mathbf{DM}^-(k)$, the higher Chow groups, motivic cohomology, Suslin homology and bivariant cycle cohomology are all representable.
Let X be a smooth scheme and let Z be a smooth subscheme of codimension c. Then we have an exact triangle:

$$M(X-Z) \longrightarrow M(X) \longrightarrow M(Z)(c)[2c].$$

Homotopy category of sheaves

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We have two different circles

$$S_s^1 = \Delta^1/\partial\Delta^1 \qquad \text{and} \qquad S_t^1 = \mathbb{A}^1 - 0.$$

Let $T = S_s^1 \wedge S_t^1$. Then the formalism of T-spectra provides a stable homotopy category $\mathcal{SH}(k)$.

What happened to transfers?

Let $k \subset L$ be a finite separable field extension and consider $x : \mathrm{Spec} L \to \mathbb{A}^1_k$. Then the Thom isomorphism gives

$$\mathbb{P}^1 - (\mathrm{Spec} L) \to \mathbb{P}^1 \to Th(\nu_i) \cong T \wedge (\mathrm{Spec} L).$$

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But

$$\begin{array}{ccccc} \mathbb{A}^1 - 0 & \longrightarrow & \mathbb{A}^1 & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1/\mathbb{A}^1 \end{array}$$

and therefore $T \cong (\mathbb{P}^1, \infty)$ and this yields a map:

$$T \to T \wedge [\mathrm{Spec} L],$$

which is the “transfer” (after inverting T).

Comparisons

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Let $\mathrm{char}\,k = 0$. Then $\mathcal{SH}(k) \otimes \mathbb{Q} \cong \mathbf{DM}(k, \mathbb{Q})$ when -1 is a sum of squares. In general $\mathbf{DM}(k)$ is a summand in $\mathcal{SH}(k)$.

À quoi servent-ils?

Milnor conjecture

Let F be a field of characteristic different from 2. Let $T^*(F)$ be the tensor algebra, i.e., $\oplus_{n\in\mathbb{N}}(F^\times)^{\otimes n}$. Let I be the ideal generated by $\{a \otimes (1-a) : a \in F^\times \setminus \{1\}\}$.
Then Milnor's K-theory is the quotient:

$$K^*_M(F) = T^*(F)/I.$$

The Galois symbol induces a map

$$K^*_M(F) \otimes \mathbb{Z}/2\mathbb{Z} \to H^*(F, \mathbb{Z}/2\mathbb{Z}).$$

Theorem (Voevodsky) The map above is an isomorphism of graded rings.

Bloch-Kato conjecture

Conjecture The map

$$K_M^*(F) \otimes \mathbb{Z}/\ell\mathbb{Z} \to H^*(F, \mathbb{Z}/\ell\mathbb{Z})$$

is an isomorphism of graded rings for $\ell \neq 2$.

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Conjecture The map

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Now, let X be a smooth scheme and let α be the projection $(Sm/k)_{ét} \to (Sm/k)_{Zar}$. Let $\underline{z}(-,*)$ be Bloch's complex, then $\underline{z}_n(-,*)_{ét} \otimes \mathbb{Z}/m \cong \mu_m^{\otimes n}$.

The Bloch-Kato conjecture is equivalent to the Beilinson-Lichtenbaum conjecture: the induced map

$$\underline{z}_n(-,*) \otimes \mathbb{Z}/m \to \tau_{\leq n} R\alpha_* \mu_m^{\otimes n}$$

is an isomorphism.

Applications to K-theory

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  almost means calculated $K_n(\mathbb{Z})$ for $n < 20000$ and $n \neq 4m (m \geq 2)$. The groups $K_{4m}(\mathbb{Z})$ are conjecturally zero (equivalent to Vandiver's conjecture).
- Weibel and Pedrini on the K-theory of surfaces and curves.

Other applications

- Kahn and Sujatha constructed (several) categories of birational motives (assuming resolution of singularities). From these, they construct new birational invariants which generalize in particular nonramified cohomology.

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- Vishik (with Orlov, Voevodsky, Rost) studied the motive of quadrics to produce new invariants and provide new information about the Witt ring of quadratic forms.

À quoi ne servent-ils pas?

Beilinson-Soulé conjecture

Conjecture Let X be a regular scheme. Then

$$H^i(X,\mathbb{Z}(n)) = 0 \begin{cases} n > 0 \text{ and } i \leq 0, \\ n = 0 \text{ and } i < 0. \end{cases}$$

From the definition, if X is smooth then $H^i(X,\mathbb{Z}(n)) = 0$ if $i > n + \dim X$. Also when $i > 2n$, because of the isomorphism with higher Chow groups.
Under Beilinson-Lichtenbaum conjecture, the Beilinson-Soulé conjecture equivalent to its counterpart with $\mathbb{Q}$ coefficients.

Lemma The Beilinson-Soulé conjecture is true for $\dim X \leq 1$ and $H^i(X,\mathbb{Z}(n)) = 0$ for $i \geq \dim X + 2$ (modulo some technical conditions).

Standard conjectures

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Let H be a Weil cohomology over a smooth projective variety X. If Y is an hyperplane section of X, let L be the morphism induced on $H^*(X)$.

A L an isomorphism.

B The morphism $*$ associated to L is algebraic.

C The components of the diagonal are algebraic.

I $< -, *- >$ is positive definite on algebraic cycles.

D Numerical equivalence is the same as homological equivalence.

Bloch-Beilinson Filtration

All with $\mathbb{Q}$ coefficients. Then there is a descending filtration $CH^*(X) = F^0 \supset F^1 \supset \ldots$ such that

1. $F^1 = \{z | z \sim_h 0\}$
2. $F^r \cdot F^s \subset F^{r+s}$
3. F^* is respected by pullbacks and pushforwards
4. $F^r CH^j / F^{r+1} CH^j$ only depends on h^{2j-r}.
5. $F^r = 0$ for $r >> 0$.

Mixed motives

There is an abelian category $\mathcal{MM}$ of mixed motives with the following properties:

1. Tannakian
2. the semisimple part is given by numerical motives
3. admits a weight filtration
4. every variety admits an associated motive and one with compact supports
5. blow-up sequences, localization and Mayer-Vietoris
6. homotopy invariant
7. dual description
8. spectral sequence for X smooth

$$\mathrm{Ext}^p(\mathbb{I}, h^q(X)(n)) \Rightarrow H^{p+q}(X, \mathbb{Q}(n))$$

The last condition implies the Beilinson-Soulé conjecture.

Bloch-Beilinson Filtration and $\mathcal{MM}$

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1. $F^1 = \{z | z \sim_h 0\}$
2. $F^r \cdot F^s \subset F^{r+s}$
3. F^* is respected by pullbacks and pushforwards
4. Assume there is a an abelian category $\mathcal{MM}$ and a functorial isomorphism:

$$F^r CH^j / F^{r+1} CH^j \simeq \mathrm{Ext}^r_{\mathcal{MM}}(1, h^{2j-r}(X)(j)).$$

5. $F^r = 0$ for $r >> 0$.

Bloch-Beilinson Filtration and $\mathcal{DM}$

For any k-scheme X there is a triangulated $\mathbb{Q}$-linear tensor category $\mathcal{DM}(X)$ with a t-structure which satisfies:

1. six-functors formalism
2. the heart has a weight filtration
3. Tate objects
4. the subcategory of semisimple objects in $\mathcal{M}(k)$ should be the Grothendieck motives.
5. There are canonical isomorphisms

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \simeq \mathrm{Hom}_{\mathcal{DM}}(\mathbb{Q}_{\mathcal{M}}, \mathbb{Q}_{\mathcal{M}}(j)[i]).$$

6. realization functors to Eckedahl motives and mixed Hodge structures.

Candidates: Voevodsky, Hanamura, Levine and Nori.
Application to filtrations of Chow groups.

Murre conjecture

Let X be a smooth projective variety. Then Murre conjectured that:

1. the decomposition of the diagonal is algebraic.
2. the correspondences $\pi_{2j+1}, \ldots, \pi_{2d}$ act as zero on $CH^j(X)_{\mathbb{Q}}$.
3. Let $F^r CH^j(X)_{\mathbb{Q}} = \mathrm{Ker}\pi_{2j} \cap \ldots \cap \mathrm{Ker}\pi_{2j-r+1}$. Then F is independent of the choice of the π_j.
4. $F^1 CH^j(X)_{\mathbb{Q}} = CH^j(X)_h$.

Theorem Murre's conjecture is equivalent to the existence of the Beilinson's filtration on Chow groups, and the two filtrations coincide.