
Welcome to
Hiroshima!

The determinant and the dual of finite dimensional motives

Shun-ichi Kimura (Hiroshima University)
木村俊一（広島大学）

Today's Goal

(1) What is finite dimensional motive good for?

(2) Motives of hypersurfaces and 1-dimensional motives

rk 1 vector bundles (i.e., line bundles) are invertible. Also, 1-dimensional vector space K is trivial (in the sense $K \otimes V \simeq V$ for any V). Analogy for motives?

(3) Some remark on Schur finite motives

X: smooth projective variety over $\mathbb{C}$

Pure motive is a pair (X, α), with

$$\alpha : X \vdash X$$

an idempotent correspondence.

For any Weil cohomology theory H, we define

$$H^*(M) := \alpha^*(H^*(X)) \subset H^*(X)$$

An additive functor F is called conservative if

$$F(M) = 0 \iff M = 0.$$

Conjecture The (singular) cohomology functor is conservative:

$$\begin{array}{ccc} & H^* & \\ \{\text{Pure Chow Motives}\} & \rightarrow & \{\text{Vector Spaces}/\mathbb{Q}\} \\ M = (X, \alpha) & \mapsto & H^*(M, \mathbb{Q}) \end{array}$$

If $H^*(M, \mathbb{Q}) = 0$, then $M = 0$.

The information of motives must be detected by their cohomology groups.

Prop. The conjecture implies Bloch's conjecture.

(proof) X: Surface with $p_g = q = 0$ (We may just assume that the cycle map $\mathrm{CH}^* X_{\mathbb{Q}} \to H^*(X, \mathbb{Q})$ is surjective.)

By Künneth, one can write the diagonal as

$$[\Delta_X] = \sum \alpha_i \times \beta_i \in H^*(X \times X, \mathbb{Q})$$

$$\begin{cases} \alpha_i = \mathrm{cl}(\tilde{\alpha}_i) \\ \beta_i = \mathrm{cl}(\tilde{\beta}_i) \end{cases} \quad \text{algebraic cycles}$$

$$M := \left(X, [\Delta_X] - \sum \tilde{\alpha}_i \times \tilde{\beta}_i \right)$$

is a motive, with $H^*(M) = 0$.

Conjecture implies $M = 0$, namely

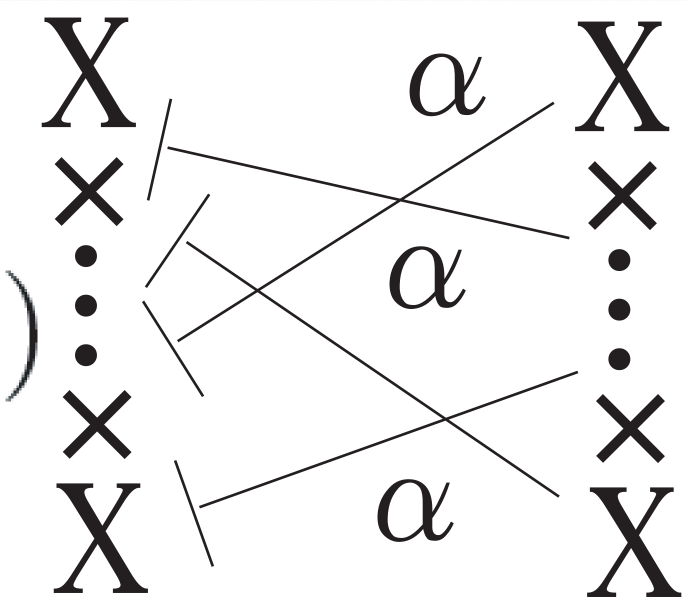
$$[\Delta_X] = \sum \tilde{\alpha}_i \times \tilde{\beta}_i \in \mathrm{CH}^*(X \times X)$$

Then $\mathrm{CH}^* X$ is generated by $\tilde{\beta}_i$'s (finitely generated!), hence representable. $\square$

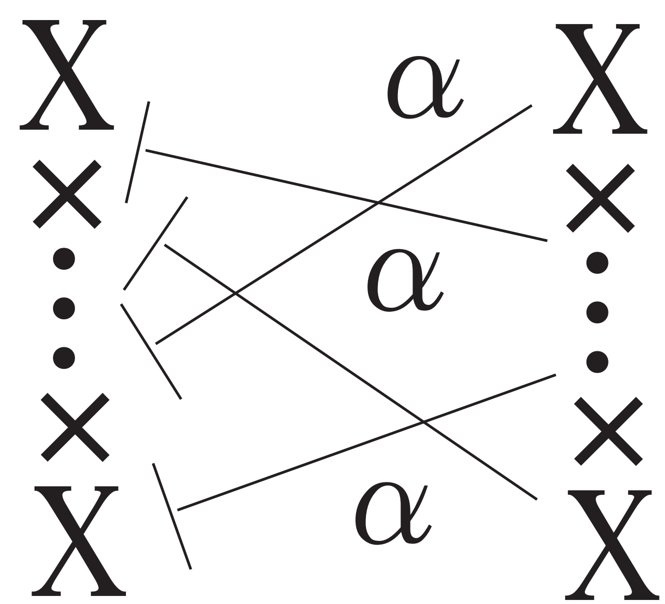
A Motive M is **finite dimensional** if $M = (X, \alpha)$ can be written as $M = M^{even} \oplus M^{odd}$ s.t. for $N \gg 0$,

$$\wedge^N M^{even} = 0, \quad \mathrm{Sym}^N M^{odd} = 0$$

$$\wedge^N M = \left(X^N, \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sgn}(\sigma) \right)$$



$$\mathrm{Sym}^N M = \left(X^N, \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \right)$$



Finite dimensionality Conjecture

Any Chow motive (X, α) is finite dimensional.

Facts (i) Motives of curves are finite dimensional (Shermenev).

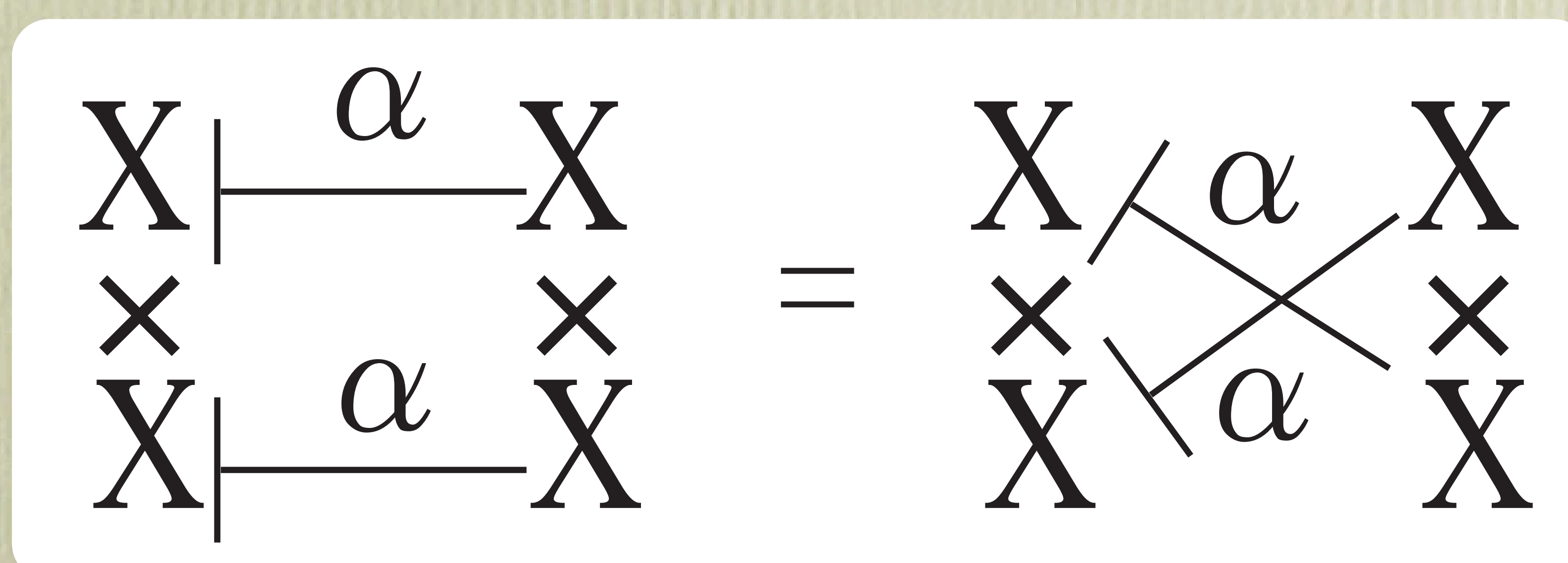
(ii) Product of finite dimensional motives are finite dimensional. (K)

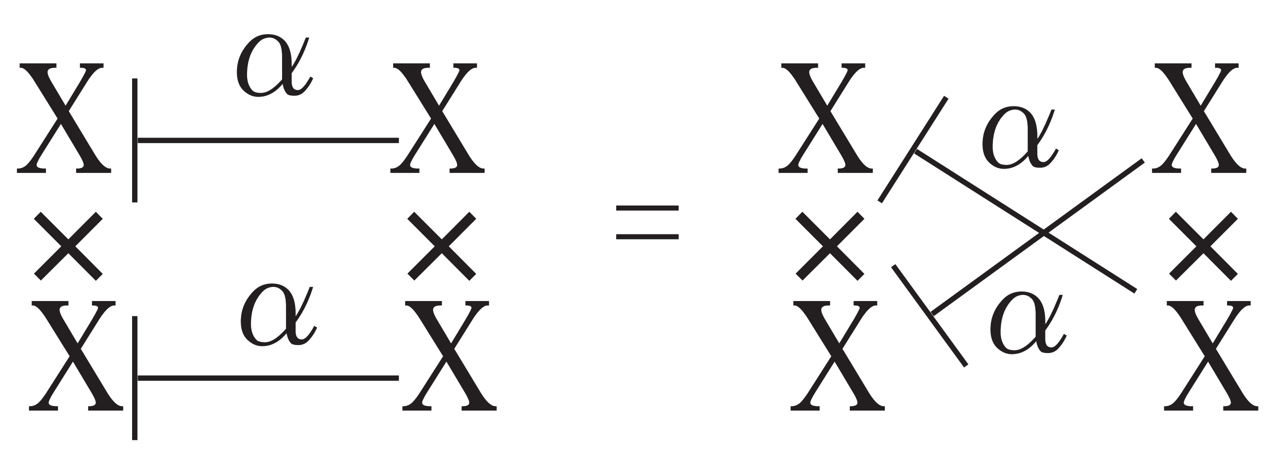
(iii)Submotives and quotient motives of finite dimensional motives are finite dimensional. (K)

Theorem If the finite dimensionality conjecture holds, then H^* is conservative.

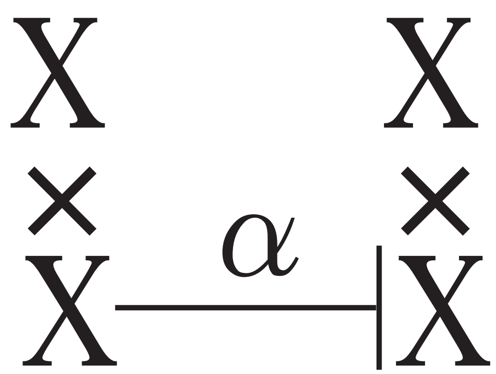
(Proof)

Case 1: When $M = (X, \alpha)$ is oddly 1-dimensional (i.e., $\wedge^2 M = 0$) and $H^*(M) = 0$

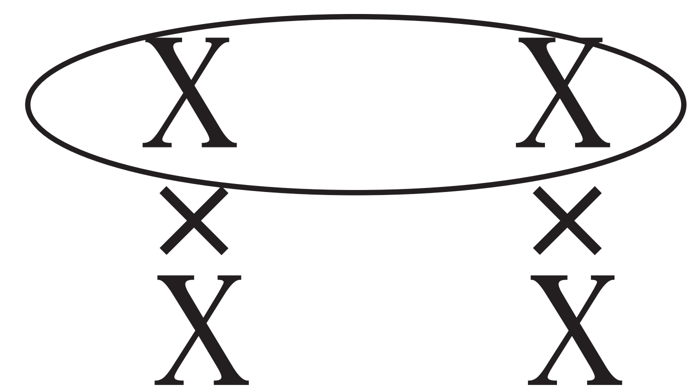




Intersect with



push-forward to X×X

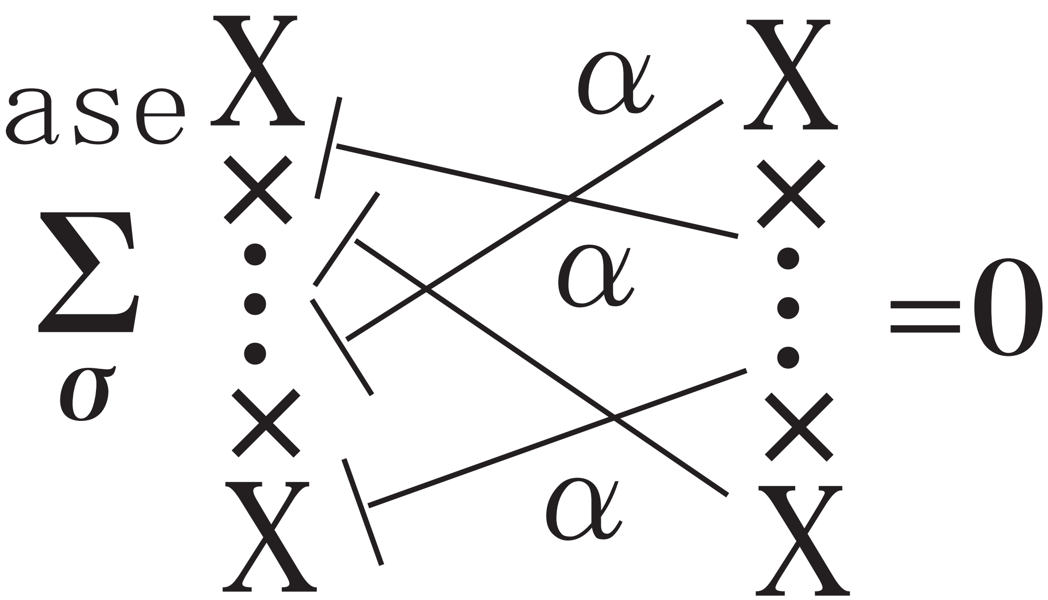


$$\text{LHS}=\alpha \cdot \deg(\alpha \cdot {}^t\alpha) = 0$$

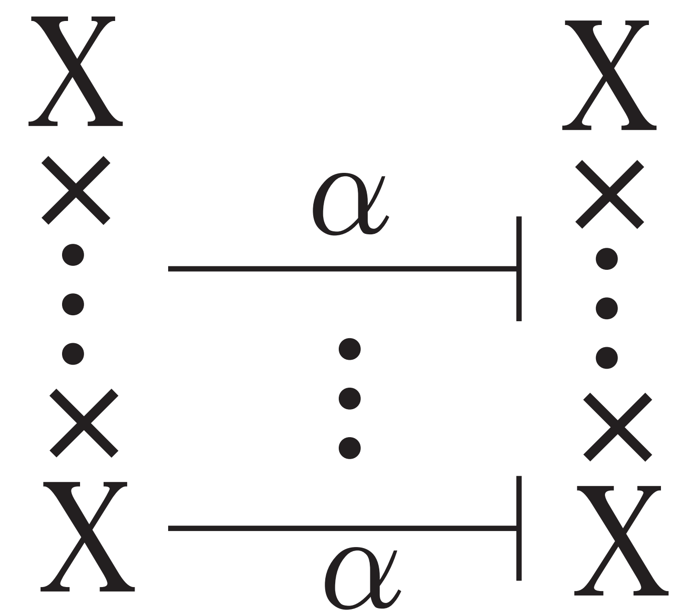
$$\text{RHS}=\alpha \circ \alpha \circ \alpha = \alpha$$

Case 2: When $M = (X, \alpha)$ is oddly (or evenly) finite dimensional, and $H^*(M) = 0$

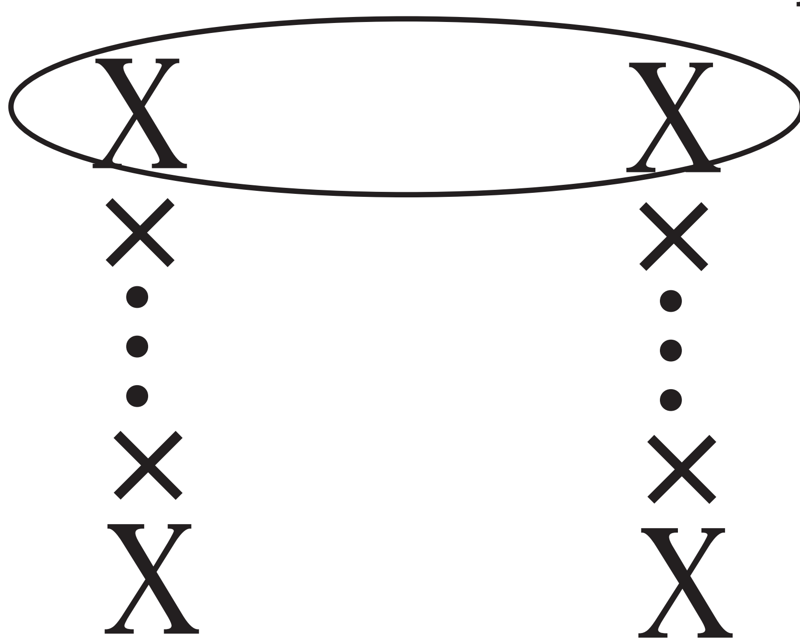
Odd case



Intersect with



push-forward to X×X



Calculation shows

$(N-1)!\,\alpha = 0$

General case: $M = M^{even} \oplus M^{odd}$ with $H^*(M) = 0$

M^{even} and M^{odd} are finite dimensional with $H^*(M^{even}) = H^*(M^{odd}) = 0$

By the previous case,

$$M^{even} = M^{odd} = 0$$

$\square$

Def. If $\wedge M^d \neq 0$ and $\wedge^{d+1} M = 0$ then $\dim M := d$. If $\mathrm{Sym}^e M \neq 0$ and $\mathrm{Sym}^{e+1} M = 0$ then $\dim M := e$. When $M = M^{even} \oplus M^{odd}$, define $\dim M := \dim M^{even} + \dim M^{odd}$.

Cor. If M is finite dimensional, then $\dim M = \dim_{\mathbb{Q}} H^*(M, \mathbb{Q})$.

(Proof) Immediate from the conservativity of H^* and

$$\begin{aligned} H^*(\mathrm{Sym}^N M^{odd}) &= \wedge^N H^*(M^{odd}) \\ H^*(\wedge^N M^{even}) &= \wedge^N H^*(M^{even}) \end{aligned} \quad \square$$

How to attack finite dimensionality?

Ayoub: "Almost" enough to prove the finite dimensionality for the motives of hypersurfaces.

We will come back to this later.

$X \subset \mathbb{P}^{n+1}$: hypersurface of degree d,
$h \in \mathrm{CH}^1 X$: hyperplane section.

$$M := \overline{X}_{prim} := \left(X, [\Delta_X] - \frac{1}{d} \sum_{i=0}^{n} h^i \times h^{n-i} \right)$$

then $H^*(M) = H^n(M)$

Assume the conservativity of H^* and let $M = \overline{X}_{prim}$.

If $n = \dim X$ is odd, then for $N \gg 0$,

$$H^*(\mathrm{Sym}^N M) = 0, \text{hence } \mathrm{Sym}^N M = 0.$$

If $n = \dim X$ is even, then

$$H^*(\wedge^N M) = 0, \text{hence } \wedge^N M = 0.$$

Also $\wedge^{top} M$ or $\mathrm{Sym}^{top} M$ is 1-dimensional.

Hope: One dimensional motives carry more geometric information!

Characterization of 1-dim. motives

Prop. (1) Nonzero motive (X, α) is isomorphic to $(P, [\Delta_P])$ (up to dimension twist) if and only if $\alpha = \beta \times \gamma \in \mathrm{CH}^*(X \times X)$ for some $\beta, \gamma \in \mathrm{CH}^* X$.

(2) Motive $M = (X, \alpha)$ is 1-dimensional if and only if M is invertible. In this case, M is evenly finite dimensional.

(3) Assuming the Hodge conjecture, M is 1-dimensional if and only if $M \simeq (P, [\Delta_P])$.

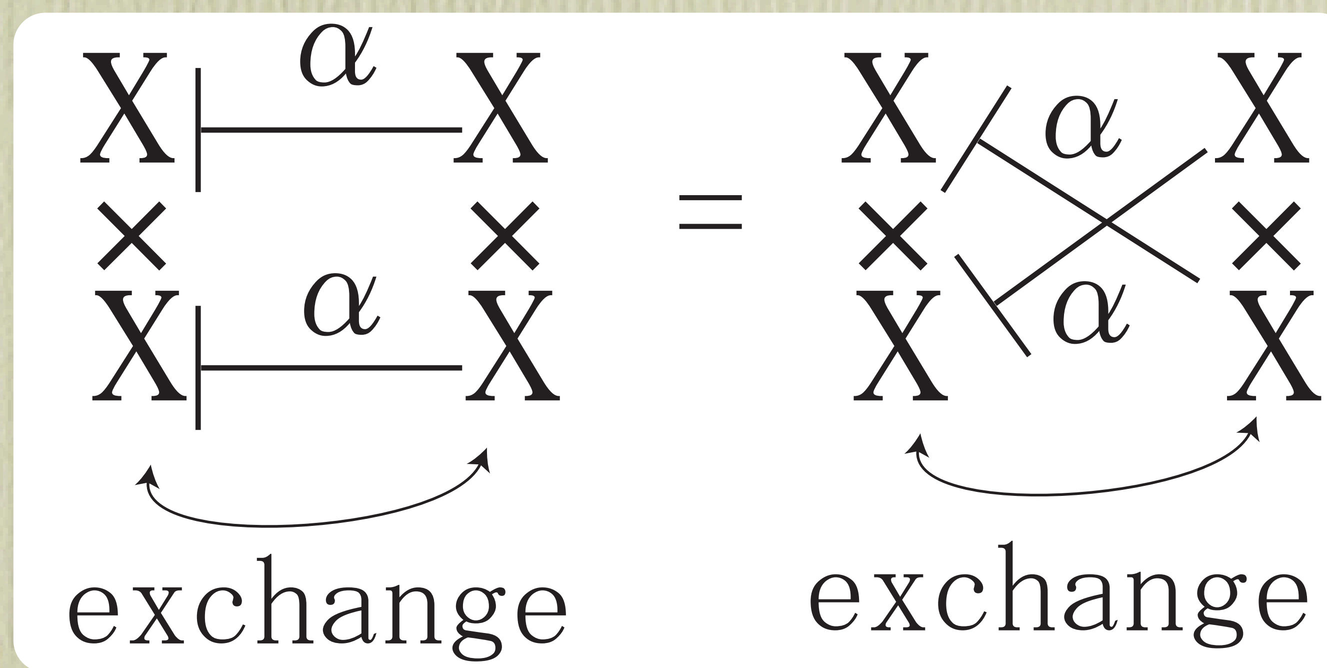
(Proof) (1) If $(X, \alpha) \simeq (P, [\Delta_P])$ then

$$\alpha = X \overset{\beta}{\vdash} P \overset{\gamma}{\vdash} X = \beta \times \gamma$$

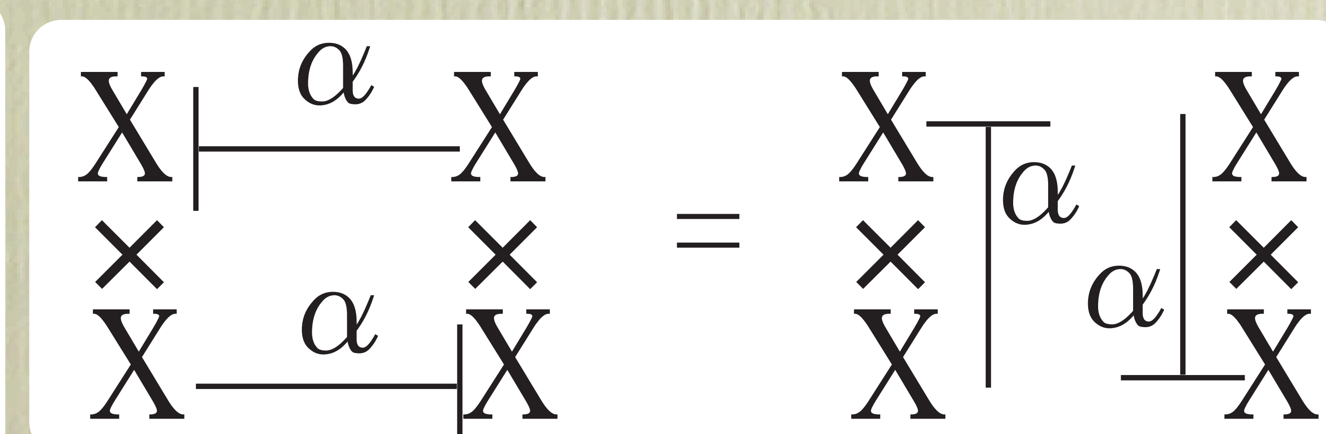
Conversely, if $\alpha = \beta \times \gamma$, then $\beta : (X, \alpha) \to (P, [\Delta_P])$ and $\gamma : (P, [\Delta_P]) \to (X, \alpha)$ are inverse to each other.

(2) If M is 1-dimensional, then $H^*(M)$ has a 1-dimensional Hodge structure, hence M is even.

If (X, α) is 1-dimensional



$$(X, \alpha) \otimes (X, {}^t\alpha) =$$



$\cong (P, [\Delta_P])$

Conversely if $(P, [\Delta_P]) \simeq M \otimes N$, taking $(\_\_)^{\otimes 2}$ both sides,

$$
\begin{aligned}
& (P, [\Delta_P]) \\
\simeq\ & M^{\otimes 2} \otimes N^{\otimes 2} \\
\simeq\ & (\mathrm{Sym}^2 M \oplus \wedge^2 M) \otimes (\mathrm{Sym}^2 N \oplus \wedge^2 N) \\
\simeq\ & (\mathrm{Sym}^2 M \otimes \mathrm{Sym}^2 N) \oplus (\mathrm{Sym}^2 M \otimes \wedge^2 N) \\
& \oplus (\wedge^2 M \otimes \mathrm{Sym}^2 N) \oplus (\wedge^2 M \otimes \wedge^2 N)
\end{aligned}
$$

3 out of 4 direct summands are 0.

Say, if $\wedge^2 M \otimes \mathrm{Sym}^2 N = \wedge^2 M \otimes \wedge^2 N = 0$, then

$$\begin{aligned} & \wedge^2 M \otimes N^{\otimes 2} \\ \simeq \; & \wedge^2 M \otimes (\mathrm{Sym}^2 N \oplus \wedge^2 N) \\ = \; & 0 \end{aligned}$$

hence $\wedge^2 M \simeq (\wedge^2 M \otimes N^{\otimes 2}) \otimes M^{\otimes 2} = 0.$

(3) If $M = (X, \alpha)$ is 1-dimensional, then $\dim H^*(M) = 1$, and is generated by a Hodge cycle $\mathrm{cl}(\beta)$. Similarly, as $M^\vee := (X, {}^t\alpha)$ is 1-dimensional, $H^*(M^\vee)$ is generated by $\mathrm{cl}(\gamma)$, then $M \simeq (X, \gamma \times \beta)$. $\square$

$\wedge^{top} M^{even}$ or $\mathrm{Sym}^{top} M^{odd}$ are 1-dimensional motives, and are called determinant of M^{even} (or M^{odd}).

Assuming the Hodge conjecture, $\wedge^{top-1} M^{even}$ or $\mathrm{Sym}^{top-1} M^{odd}$ are the dual motives of M^{even} (or M^{odd}).

PROBLEM Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface, $M := \overline{X}_{prim}$, assuming finite dimensionality and Hodge conjecture,

$$\left.\begin{array}{ll} (n \text{ even}) & \wedge^{top} M \\ (n \text{ odd}) & \mathrm{Sym}^{top} M \end{array}\right\} = (X^e, c(\alpha \times \alpha))$$

where $e = \dim H^*(M)$, $c \in \mathbb{Q}$, $\alpha \in \mathrm{CH}^{ne/2}(X^e)$.

(1) Find such α (future research plan).

(2) Does such α imply the finite dimensionality of M?

If $n = \dim X$ is even,

$$\wedge^2 (\wedge^e M) \twoheadrightarrow \wedge^{2e} M$$

so M is evenly finite dimensional.

But if $n = \dim X$ is odd, $\wedge^2(\mathrm{Sym}^e M) = 0$ does not imply $\mathrm{Sym}^{2e} M = 0$.
For $M = (P, [\Delta_P])$, $\wedge^2(\mathrm{Sym}^e M) = 0$ but $\mathrm{Sym}^N M \neq 0$ for any $N > 0$.

Schur Finite Motives
(Carlo Mazza, P. Deligne)

Each Young tableau λ determines the Young symmetralizer

$$c_\lambda = \sum c_\lambda(\sigma) \cdot [\sigma] \in \mathbb{Q}[\mathfrak{S}_n]$$

For $M = (X, \alpha)$, define

$$S_\lambda(M) := \left(X^n, \sum_\sigma c_\lambda(\sigma) \prod_{i=1}^{n} \pi^*_{i,\sigma(i)} \alpha \right)$$

M is Schur finite if $S_\lambda M = 0$ for some λ.

Young Tableau

$$\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

$$\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline\end{array} \Longrightarrow S_\lambda M = \mathrm{Sym}^n M$$

$$\lambda = \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline\end{array} \Longrightarrow S_\lambda M = \wedge^n M$$

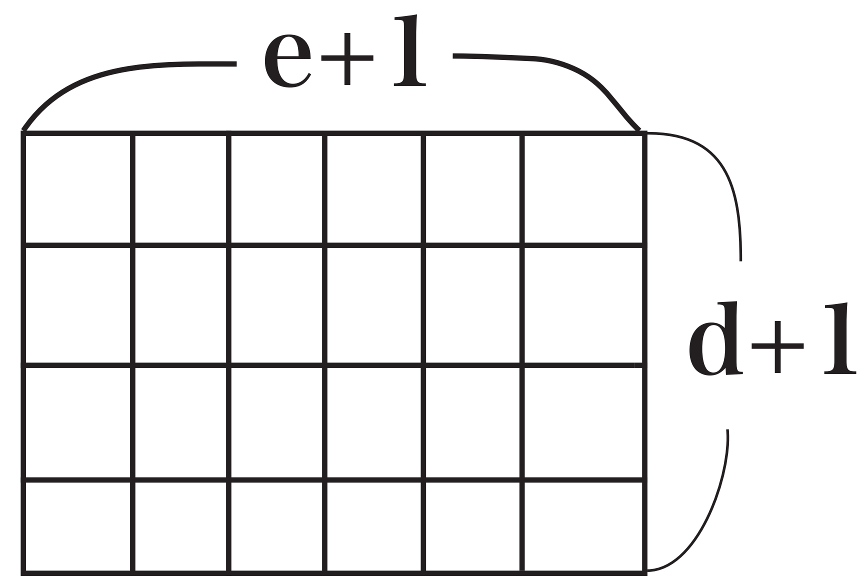
Schur Finite conjecture

For any M, $S_\lambda M = 0$ for some λ.

Fact If $M = M^{even} \bigoplus M^{odd}$

$$\begin{cases} \dim M^{even} & = & d \\ \dim M^{odd} & = & e \end{cases}$$

$\Rightarrow S_\lambda M = 0 \iff \lambda \supset$

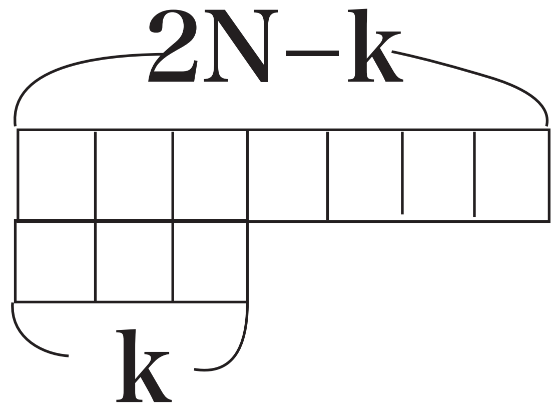


Finite dimensional motives are Schur finite.

Ayoub proves that if the motives of hypersurfaces are Schur finite, then all the motives are Schur finite.

Guletskii "proves" that Schur finite implies Conservativity of cohomology functor.

$$\wedge^2(\mathrm{Sym}^N M) = \bigoplus_{\substack{0 < k \le N \\ k\,:\,\mathrm{odd}}} S_{\lambda_k} M$$

$\lambda_{\mathbf{k}} =$ 

(Littlewood)

M with $\wedge^2(\mathrm{Sym}^N M) = 0$ is Schur finite. By Guletskii, conservativity of cohomology functor for such motives follows.

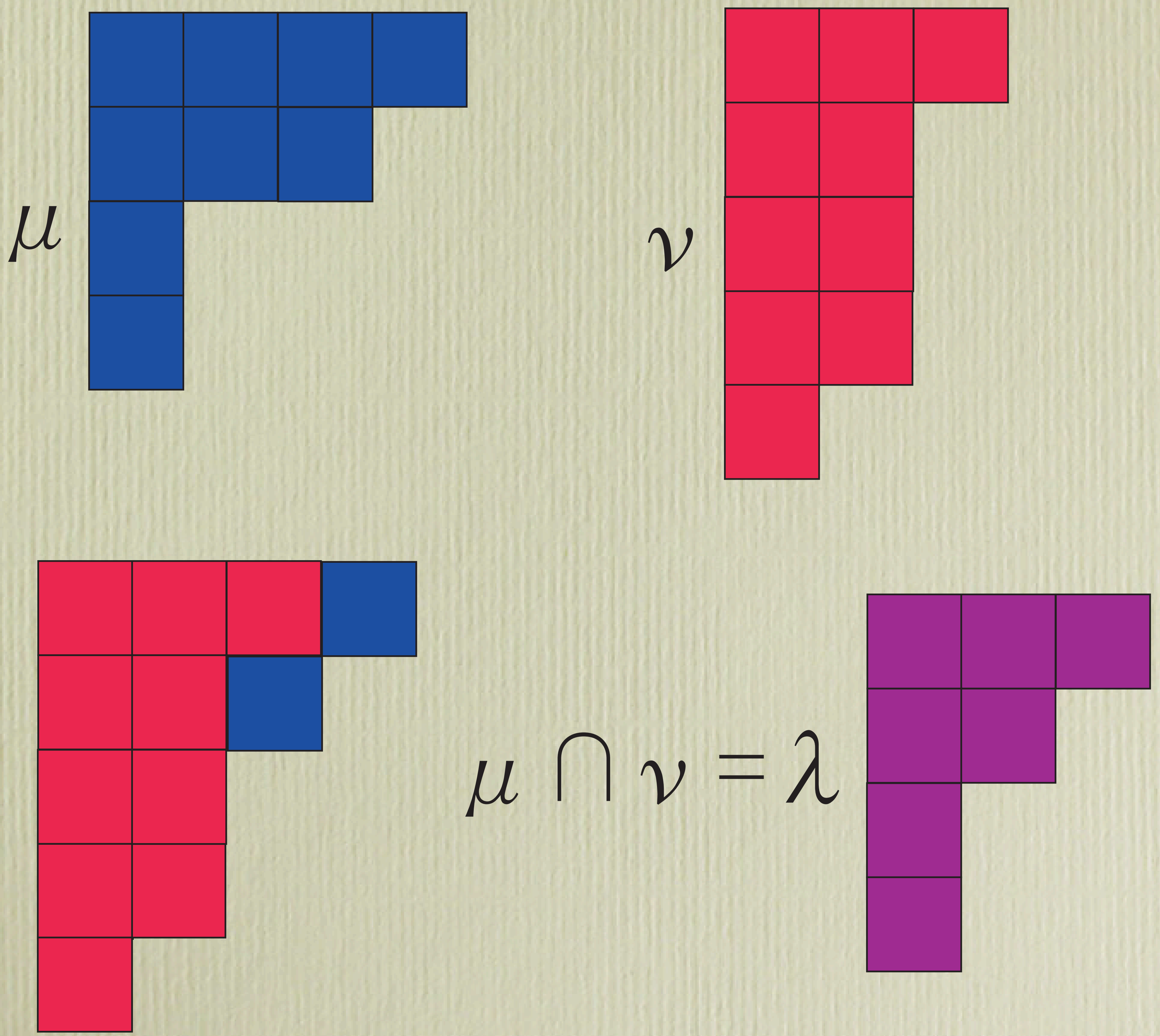
Question Let M be a Schur finite element of a tensor category. What can you say about the set $\{\lambda | S_\lambda M = 0\}$?

Theorem In a tensor category where $M^{\otimes n} = 0$ implies $M = 0$ (e.g., the motive of pure Chow motives), if M is Schur finite, then there exists a Young diagram λ such that $S_\mu M = 0$ if and only if $\lambda \subset \mu$.

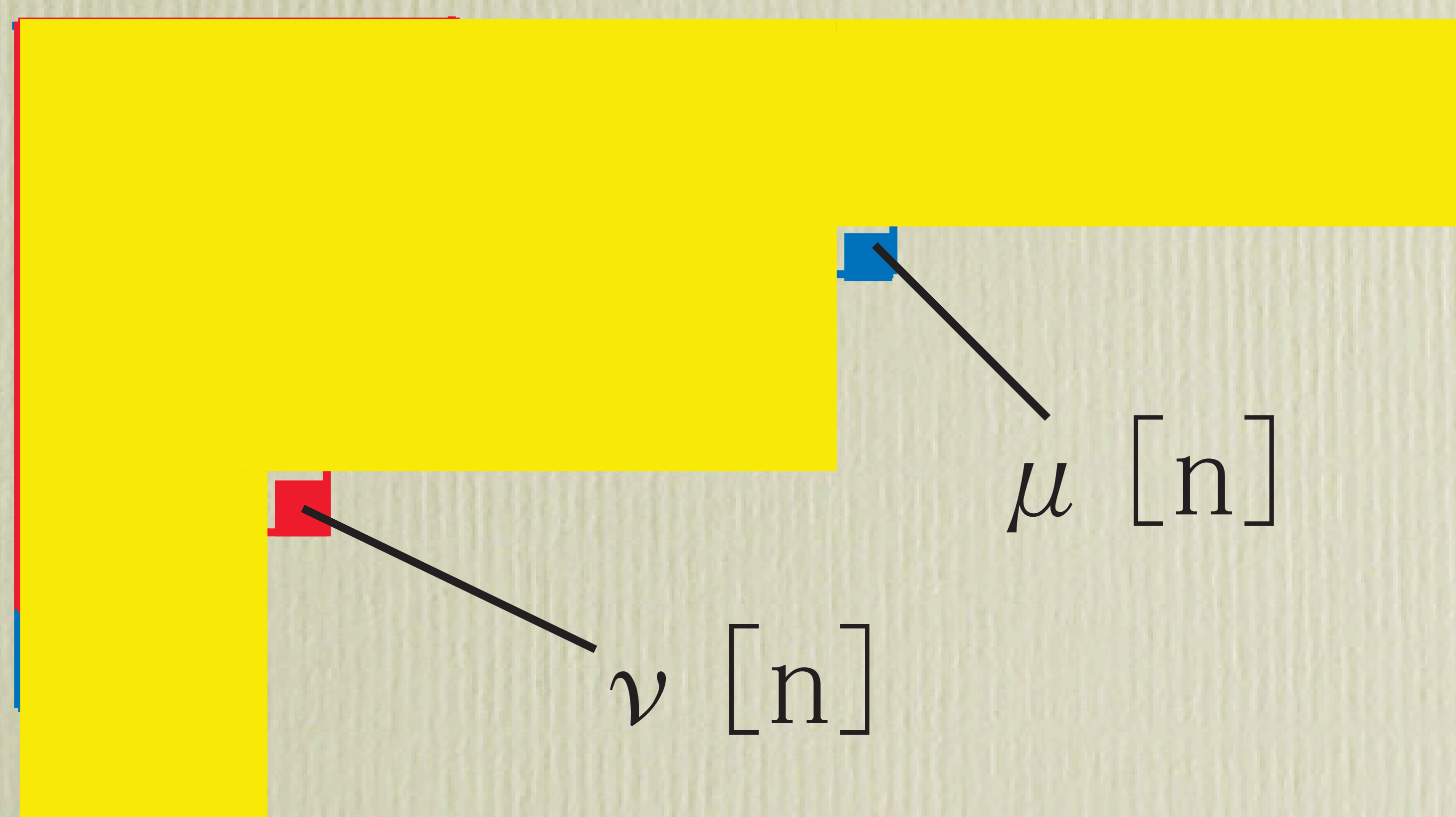
(Proof) If $\mu \supset \lambda$ and μ has m boxes, λ has ℓ boxes, then $S_\lambda M \otimes M^{\otimes(m-\ell)}$ contains $S_\mu M$ as a direct summand. So $S_\lambda M = 0$ implies $S_\mu M = 0$.

Enough to show that if $S_\mu M = 0$ and $S_\nu M = 0$, then for $\lambda := \mu \cap \nu$, $S_\lambda M = 0$. We are reduced to the following lemma.

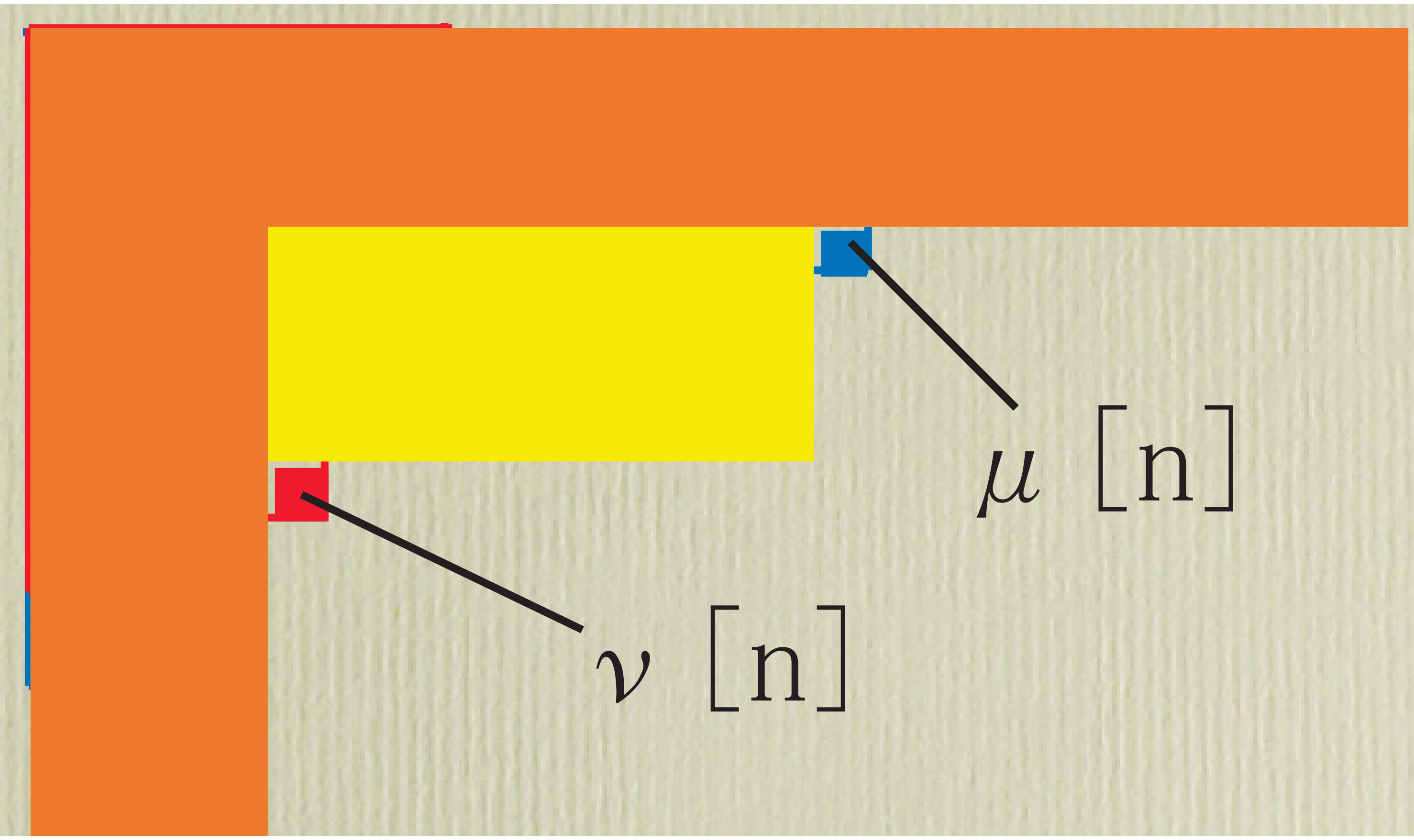
Lemma Each direct summand of $(S_\lambda M)^{\otimes N}$ either containes μ or ν for $N \gg 0$.


μ
ν
μ ∩ ν = λ

(Proof of Lemma) Assume NOT. For each $n > 0$, there is a direct summand $S_{\tau_n} M$ of $(S_\lambda M)^{\otimes n}$ which misses the box $\mu[n] \in \mu$ and $\nu[n] \in \nu$. We may assume that $\mu[n]$ (resp. $\nu[n]$) are fixed for all n. To avoid these boxes, τ_n are confined to the yellow area.



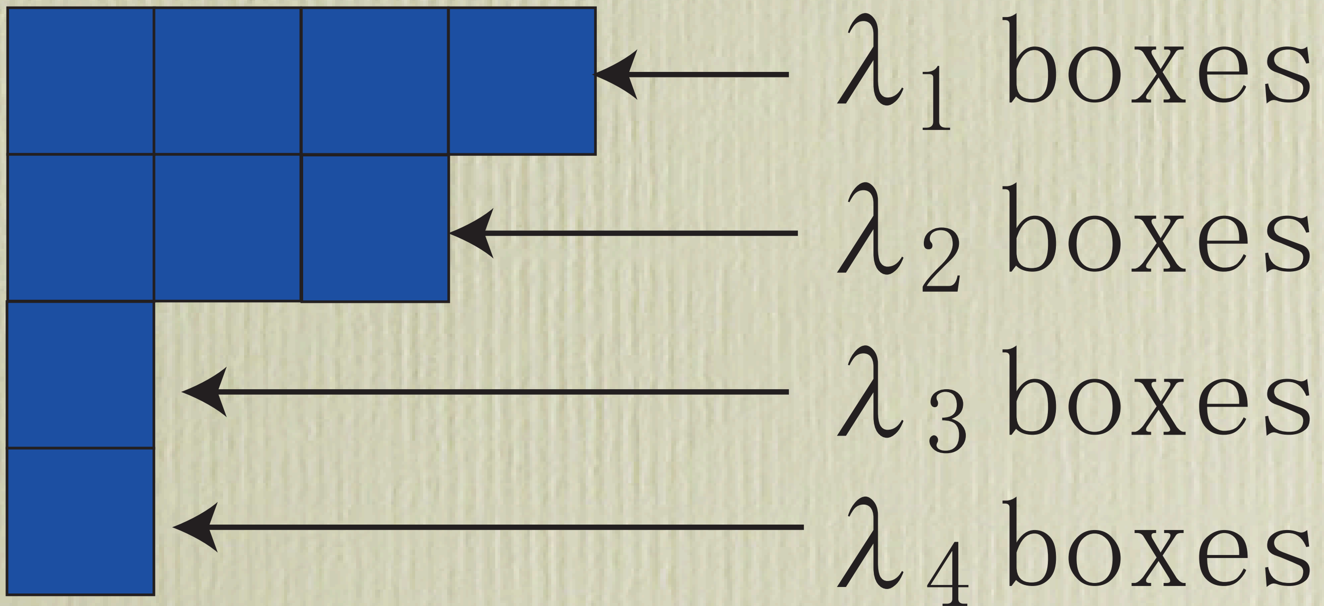
Claim When we tensor $S_\lambda M$ by Littlewood-Richardson rule, at least one new box is outside the orange area.



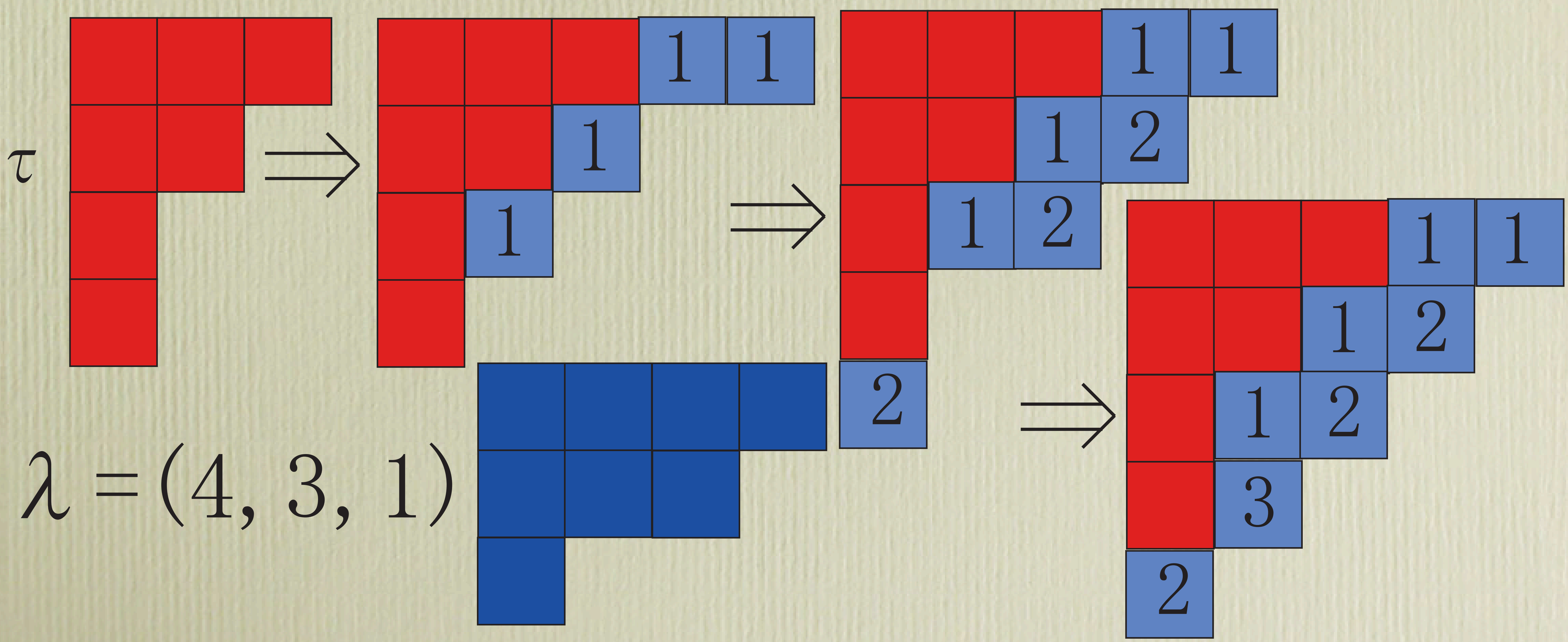
There are only finitely many boxes in the yellow area outside the orange area. So assuming the claim, $S_{\tau_n} M$ flows out of the yellow area for $n \gg 0$.

Littlewood-Richardson Rule Write $\lambda = (\lambda_1, \ldots, \lambda_k)$ as below.

$$\lambda = (4, 3, 1, 1)$$



To tensor $S_\lambda M$ to $S_\tau M$, First add λ_1 boxes with number 1 to enlarge the Young diagram, and add λ_2 boxes with number 2 to further enlarge the Young diagram, and so on.

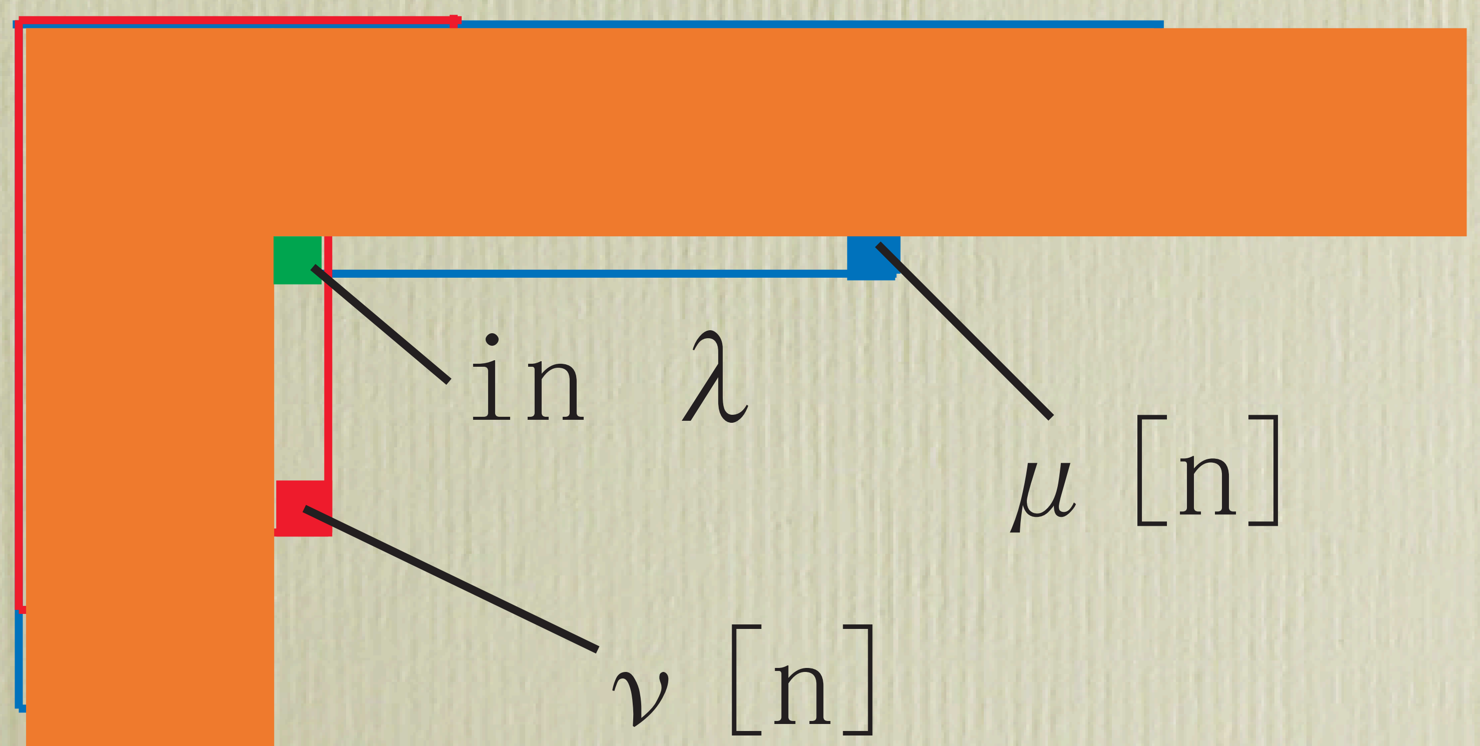


Rule:

(1) Same numbered boxes are not in the same column.

(2) Start counting the number of boxes from upper right corner, to the left, and then down. While counting, if $i < j$, then i-numbered boxes are always no less than j numbered boxes.

Let $\mu[n]$ and $\nu[n]$ as below. $\mu[n]$ is in the k-th row, and $\nu[n]$ is in the ℓ-th column, then $\lambda_k \geq \ell$. By Rule (1), one of the k-numbered box is more right than the $\ell - 1$-st column. By Rule (2), all the k-numbered boxes are below $k - 1$st row. We are done.



Application: If $\wedge^2(\mathrm{Sym}^N M) = 0$, then $S_\lambda M = 0$ with

$$\lambda = \begin{cases} (N+1,1) & (N \text{ is even}) \\ (N,1) & (N \text{ is odd}) \end{cases}$$

In the case of pure motives, by the theorem of Guletskii, the minimal vanishing λ is a rectangle, so either $\lambda^2 M = 0$ or $\mathrm{Sym}^{N+1} M = 0$.

Question Characterize the tensor category where the minimal vanishing λ is rectangle.


Thank you very much!