p-adic Étale Tate Twists and Arithmetic Duality

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Introduction.

X : a projective smooth variety over a finite field $k = \mathbb{F}_q$

p $:= \mathrm{ch}(k)$

$\zeta(X,s)$: Hasse-Weil zeta function of X

$$:= \prod_{x \in X_0} (1 - N(x)^{-s})^{-1},$$

where $X_0 :=$ the set of closed points on X, and $N(x) := \sharp(\kappa(x))$.

Then we have the determinant presentation ($d := \dim(X)$, $\ell \neq p$):

$$\zeta(X,s) = \prod_{i=0}^{2d} \det(1 - \mathrm{F}^* \cdot q^{-s};\ \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell))^{(-1)^{i+1}}$$

(F^* : geometric Frobenius operator)

$$= \frac{P^1_\ell(q^{-s}) \cdot P^3_\ell(q^{-s}) \cdots P^{2d-1}_\ell(q^{-s})}{P^0_\ell(q^{-s}) \cdot P^2_\ell(q^{-s}) \cdots P^{2d-2}_\ell(q^{-s}) \cdot P^{2d}_\ell(q^{-s})}$$

with $P^i_\ell(T) := \det(1 - \mathrm{F}^* \cdot T;\ \mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell))$.

- $\zeta(X,s)$ is a rational function in q^{-s} with $\mathbb{Q}$-coefficients, because we have $\mathbb{Q}_\ell(T) \cap \mathbb{Q}((T)) = \mathbb{Q}(T)$.
- The reciprocal roots of $P^i_\ell(T)$ are algebraic integers whose complex absolute values are $q^{i/2}$. Hence $P^i_\ell(T)$ is independent of $\ell \neq p$ and belongs to $\mathbb{Z}[T]$. (Deligne)

Theorem 1. (Artin-Tate/Milne/Bayer-Neukirch/Schneider)

Let n be an integer. Assume the semi-simplicity of F^* *on*

$$\mathrm{H}^{2n}_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell) \ \textit{for all} \ \ell \neq p, \qquad \textit{and} \qquad \mathrm{H}^{2n}_{\text{cris}}(X).$$

Let ρ_n be the (positive) integer such that $\zeta(X,s)\cdot(1-q^{n-s})^{\rho_n}$ tends to a non-zero constant as $s \to n$. Then the limit

$$\lim_{s\to n} \ \zeta(X,s)\cdot(1-q^{n-s})^{\rho_n}$$

is written in terms of the cohomology groups

$$\mathrm{H}^i_{\text{ét}}(X, \widehat{\mathbb{Z}}(n)) := \prod_{\textit{all } \ell} \ \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}_\ell(n))$$

with $i = 1, 2, \ldots, 2d+1$, and a certain regulator term.

($\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}_\ell(n))$'s will be explained below.)

Problem:

- *Construct $\{\mathrm{H}^i_{\text{ét}}(X, \widehat{\mathbb{Z}}(n))\}_{i,n}$ for arithmetic schemes X.*
- *Find zeta-value formulae for arithmetic schemes.*

Results:

X : regular arithmetic scheme (with some assumption),

- Definition of the coefficient $\mathfrak{T}_r(n)_X \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ playing the role of $\mathbb{Z}/p^r\mathbb{Z}(n)$.
- Duality for $\mathrm{H}^i_{\text{ét}}(X, \mathfrak{T}_r(n)_X)$ (in case X is proper).

§1 Reveiw on étale cohomology groups over a field

- étale topology = algebraic analogue of analytic topology on complex manifolds

Example. For a variety X over $\mathbb{C}$ and a torsion sheaf $\mathcal{F}$ on $X_{\text{ét}}$,

$$\mathrm{H}^*_{\text{ét}}(X, \mathcal{F}) \simeq \mathrm{H}^*_{\text{an}}(X(\mathbb{C})^{\text{an}}, \mathcal{F}|_{X(\mathbb{C})^{\text{an}}}).$$

- étale over a field k = finite separable over k

Example. (1) For a sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{\text{ét}}$,

$$\mathrm{H}^*_{\text{ét}}(\mathrm{Spec}(k), \mathcal{F}) \simeq \mathrm{H}^*_{\text{Gal}}(G_k, \mathcal{F}(k^{\text{sep}})).$$

(2) For a variety X over k and a sheaf $\mathcal{F}$ on $X_{\text{ét}}$,

G_k naturally acts on $\mathrm{H}^*_{\text{ét}}(X_{\overline{k}}, \mathcal{F}|_{X_{\overline{k}}})$ and $^\exists$ spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\text{Gal}}(G_k, \mathrm{H}^q_{\text{ét}}(X_{\overline{k}}, \mathcal{F}|_{X_{\overline{k}}})) \Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F})$$

(Hochshild-Serre spectral sequence)

Definition 2. Let n be an integer.

(1) X scheme

m positive integer invertible on X

μ_m étale sheaf of mth roots of unity on X

We define:

$$\mathbb{Z}/m\mathbb{Z}(n) := \begin{cases} \mu_m^{\otimes n} & (n \geq 0) \\ \mathcal{H}om(\mu_m^{\otimes(-n)}, \mathbb{Z}/m\mathbb{Z}) & (n < 0). \end{cases}$$

(2) k perfect field of characteristic $p > 0$

X smooth variety over k

$W_r\Omega^n_{X,\log}$ logarithmic part of $W_r\Omega^n_X$ on $X_{\text{ét}}$ (Illusie)

Note: • $W_r\Omega^n_{X,\log} := 0$, if $n < 0$ or $\dim(X) < n$.

• $W_r\Omega^n_{X,\log}$ is a flat $\mathbb{Z}/p^r\mathbb{Z}$-sheaf and

$0 \to W_{r-1}\Omega^n_{X,\log} \to W_r\Omega^n_{X,\log} \to W_1\Omega^n_{X,\log} \to 0$ (exact).

• $W_1\Omega^n_{X,\log} \simeq \Omega^n_{X,\log} := \mathrm{Im}(d\log : (\mathcal{O}_X^\times)^{\otimes n} \longrightarrow \Omega^n_{X/k})$.

Then we define

$$\mathbb{Z}/p^r\mathbb{Z}(n) := W_r\Omega^n_{X,\log}[-n].$$

(3) In the situation of (2), the group $\mathrm{H}^i_{\text{ét}}(X, \widehat{\mathbb{Z}}(n))$ is defined as

$$\varprojlim_{(m,p)=1} \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}(n)) \times \varprojlim_{r\geq 1} \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/p^r\mathbb{Z}(n)).$$

Remark. For a projective smooth variety X over a finite field k, the group $\mathrm{H}^i_{\text{ét}}(X, \widehat{\mathbb{Z}}(n))$ is finite for $i \neq 2n, 2n+1$, and the group $\mathrm{H}^{2n}_{\text{ét}}(X, \widehat{\mathbb{Z}}(n))_{\text{tors}}$ is finite. The group $\mathrm{H}^{2n+1}_{\text{ét}}(X, \widehat{\mathbb{Z}}(n))_{\text{tors}}$ is finite as well under the semi-simplicity assumption in Theorem 1.

In what follows, we restrict our attentions to finite coefficient cases and review duality facts for varieties over finite fields.

$$X \quad : \quad \text{smooth variety over a finite field } k$$

$$d \quad := \quad \dim(X)$$

Theorem 3. (Poincaré-Pontryagin duality)

Let m be a positive integer prime to $\mathrm{ch}(k)$. *Then:*

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}_c^{2d+1}(X, \mu_m^{\otimes d}) \simeq \mathbb{Z}/m\mathbb{Z}$.

(2) *For a constructible $\mathbb{Z}/m\mathbb{Z}$-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the pairing*

$$\mathrm{H}_c^i(X, \mathcal{F}) \times \mathrm{Ext}^{2d+1-i}_{X,\mathbb{Z}/m\mathbb{Z}}(\mathcal{F}, \mu_m^{\otimes d}) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/m\mathbb{Z}$-modules, for $^{\forall}i$.

Here, $\mathcal{F}$ constructible $:=$ $^{\exists}$ finite partition $X = \cup_j Z_j$

by locally closed subschemes *such that*

$\mathcal{F}|_{Z_j}$ is locally constant with finite stalks

Corollary 4. *For any integers n and i, the pairing*

$$\mathrm{H}_c^i(X, \mu_m^{\otimes n}) \times \mathrm{H}_{\text{ét}}^{2d+1-i}(X, \mu_m^{\otimes d-n}) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/m\mathbb{Z}$-modules.

Proof. We have

$$\begin{aligned}\mathrm{Ext}^*_{X,\mathbb{Z}/m\mathbb{Z}}(\mu_m^{\otimes n}, \mu_m^{\otimes d}) &\simeq \mathrm{Ext}^*_{X,\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mu_m^{\otimes d-n}) \\ &\simeq \mathrm{H}^*_{\text{ét}}(X, \mu_m^{\otimes d-n}).\end{aligned}$$

□

Theorem 5. (Milne duality)

Let r be a positive integer. Suppose X to be proper. Then:

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}^{d+1}_{\text{ét}}(X, W_r\Omega^d_{X,\log}) \simeq \mathbb{Z}/p^r\mathbb{Z}$.

(2) *For any integers n and i, the pairing*

$$\mathrm{H}^i_{\text{ét}}(X, W_r\Omega^n_{X,\log}) \times \mathrm{H}^{d+1-i}_{\text{ét}}(X, W_r\Omega^{d-n}_{X,\log}) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules.

Key point: Construction of the trace map

(then reduced to the case $r = 1$ and the Serre duality

by the theory of Cartier operators).

Theorem 6. (Moser)

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}^{d+1}_{c}(X, W_r\Omega^d_{X,\log}) \simeq \mathbb{Z}/p^r\mathbb{Z}$.

(2) *For a constructible $\mathbb{Z}/p^r\mathbb{Z}$-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the pairing*

$$\mathrm{H}^i_c(X, \mathcal{F}) \times \mathrm{Ext}^{d+1-i}_{X,\mathbb{Z}/p^r\mathbb{Z}}(\mathcal{F}, W_r\Omega^d_{X,\log}) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules, for $^{\forall}i$.

Note:

- Theorem 5 (2) does not immediately imply Theorem 6.
- Theorem 6 recovers Theorem 5 (2) only in case $n = 0$.

Combining Theorem 3–Theorem 6, we obtain:

Fact 7.

Let m be a positive integer. Then:

(1) $\exists$ *canonical trace map* $\mathrm{H}_c^{2d+1}(X, \mathbb{Z}/m\mathbb{Z}(d)) \simeq \mathbb{Z}/m\mathbb{Z}$.

(2) *For a constructible $\mathbb{Z}/m\mathbb{Z}$-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the pairing*

$$\mathrm{H}_c^i(X, \mathcal{F}) \times \mathrm{Ext}^{2d+1-i}_{X, \mathbb{Z}/p^r\mathbb{Z}}(\mathcal{F}, \mathbb{Z}/m\mathbb{Z}(d)) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/m\mathbb{Z}$-modules, for $\forall i$.

(3) *In case X is proper, for any integers n and i, the pairing*

$$\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}(n)) \times \mathrm{H}^{2d+1-i}_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}(d-n)) \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/m\mathbb{Z}$-modules.

Remark. For a map $f : X \to X'$ of smooth varieties over k, $\exists$ a canonical pull-back map

$$f^*\mathbb{Z}/m\mathbb{Z}(n)_{X'} \longrightarrow \mathbb{Z}/m\mathbb{Z}(n)_X.$$

It is an isomorphism in the following cases (but *not*, otherwise):

- f is étale,
- $(m, \mathrm{ch}(k)) = 1$,
- $n \leq 0$,
- $\max\{\dim(X), \dim(X')\} < n$.

§2 Construction of p-adic étale Tate twists

p : prime number, r : positive integer

A : algebraic integer ring or p-adic integer ring

X : integral regular scheme flat of finite type over $\mathrm{Spec}(A)$,

satisfying:

($*$) X is smooth or a semistable family over $\mathrm{Spec}(A)$ around the fibers of characteristic p.

For $n \geq 0$, we want $\mathcal{K} \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ satisfying (T1)–(T4) (=variant of Beilinson-Lichtenbaum axioms for '$\mathbb{Z}(n)$' on $X_{\text{ét}}$):

T1: $^\exists\, t : \mathcal{K}|_V \simeq \mu_{p^r}^{\otimes n}$, *where* $V := X[1/p]$.

T2: $\mathcal{K}$ *is concentrated in* $[0, n]$.

T3: *For a locally closed regular subscheme* $i : Z \to X$ *of characteristic* p*, we have a Gysin isomorphism:*

$$\mathrm{Gys}_i^n : W_r\Omega_{Z,\log}^{n-c}[-n-c] \xrightarrow{\simeq} \tau_{\leq n+c} Ri^!\mathcal{K} \quad (c := \mathrm{codim}_X(Z))$$

in $D^b(Z_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$*, where* $W_r\Omega_{Z,\log}^{n-c} := 0$ *if* $n < c$.

T4: *A compatibility property between Gysin maps and boundary maps of Galois cohomology groups.*

Precise statement of **T4:**

Consider points y and x on X satisfying

$\mathrm{ch}(x) = p, \quad x \in \overline{\{y\}} \quad$ and $\quad \mathrm{codim}_X(x) = \mathrm{codim}_X(y) + 1.$

Put $c := \mathrm{codim}_X(x)$. (Note that $\mathrm{ch}(y) = 0$ or p.)

Then the following diagram commutes up to a sign depending only on $(\mathrm{ch}(y), c)$:

$$\begin{array}{ccc}
\mathrm{H}^{n-c+1}_{\text{ét}}(y, \mathbb{Z}/p^r\mathbb{Z}(n-c+1)) & \xrightarrow{\partial^{\mathrm{val}}} & \mathrm{H}^{n-c}_{\text{ét}}(x, \mathbb{Z}/p^r\mathbb{Z}(n-c)) \\
\Big\downarrow{\scriptstyle \mathrm{Gys}^n_{i_y}} & & \Big\downarrow{\scriptstyle \mathrm{Gys}^n_{i_x}} \\
\mathrm{H}^{n+c-1}_{y,\text{ét}}(\mathrm{Spec}(\mathcal{O}_{X,y}), \mathcal{K}) & \xrightarrow{\delta^{\mathrm{loc}}} & \mathrm{H}^{n+c}_{x,\text{ét}}(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathcal{K}),
\end{array}$$

where

∂^{val} : boundary map of Galois cohomology groups (Kato)

δ^{loc} : boundary map of a localization sequence

i_x : $x \to \mathrm{Spec}(\mathcal{O}_{X,x})$

i_y : $y \to \mathrm{Spec}(\mathcal{O}_{X,y})$

$\mathrm{Gys}^n_{i_x}$: Gysin map coming from (T3)

$\mathrm{Gys}^n_{i_y}$: $\begin{cases} \text{Gysin map coming from (T3),} & \text{if } \mathrm{ch}(y) = p, \\ \text{Gysin map coming from (T1) and} & \\ \qquad \text{Deligne's cycle class,} & \text{if } \mathrm{ch}(y) = 0. \end{cases}$

Theorem I. *For a fixed* $n \geq 0$, $^{\exists!}$ *a pair of*

$$\mathcal{K} \in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}) \quad \textit{and}$$

$$t : \mathcal{K}|_V \simeq \mu_{p^r}^{\otimes n}$$

that satisfies **T2**–**T4** *as above.*

Definition. (originally due to Schneider in the smooth case)
For $n \geq 0$, fix a pair $(\mathcal{K}, t)$ as in Theorem 1,

and define $\mathfrak{T}_r(n)_X := \mathcal{K}$ $(\in D^b(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z}))$.

For $n < 0$, define $\mathfrak{T}_r(n)_X := j_! \mathcal{H}om\left(\mu_{p^r}^{\otimes(-n)}, \mathbb{Z}/p^r\mathbb{Z}\right)$.

Remark:

- For $n > d := \dim(X)$, t induces $\mathfrak{T}_r(n)_X \simeq Rj_*\mu_{p^r}^{\otimes n}$.
- If $X \longrightarrow \operatorname{Spec}(A)$ is smooth around Y(:= union of the fibers of characteristic p) and $p \geq n+2$, then $\mathfrak{T}_r(n)_X|_Y$ is isomorphic to the syntomic complex '$\mathcal{S}_r(n)$' of Fontaine-Messing (by a result of Kurihara).
- $\mathfrak{T}_r(n)_X$ is *not* a log syntomic complex of Kato-Tsuji for $1 \leq n \leq \dim(X)$, because the latter object is rather $\tau_{\leq n} Rj_*\mu_{p^r}^{\otimes n}$.
- For the étale sheafification $\mathbb{Z}(n)_X^{\text{ét}}$ of the cycle complex defining higher Chow groups, one can hope $\mathbb{Z}(n)_X^{\text{ét}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r\mathbb{Z} \simeq \mathfrak{T}_r(n)_X$. However, to prove this, we need to show **T2** and **T3** for the left hand side.

Theorem II. (Product and Contravariant functoriality)

(1) *For integers* m *and* n, $\exists!$ *a morphism*

$$\mathfrak{T}_r(m)_X \otimes^{\mathbb{L}} \mathfrak{T}_r(n)_X \longrightarrow \mathfrak{T}_r(m+n)_X.$$

in $D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ *extending the natural isomorphism*

$$\mu_{p^r}^{\otimes m} \otimes \mu_{p^r}^{\otimes n} \xrightarrow{\simeq} \mu_{p^r}^{\otimes m+n} \quad \text{on} \quad V = X[1/p].$$

(2) *For a map* $f : X \to X'$ *of* Spec(A)*-schemes satisfying* $(*)$, $\exists!$ *a morphism*

$$\mathrm{res}^f : f^*\mathfrak{T}_r(n)_{X'} \longrightarrow \mathfrak{T}_r(n)_X.$$

in $D^-(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$ *extending the natural isomorphism*

$$g^*\mu_{p^r,V'}^{\otimes n} \xrightarrow{\simeq} \mu_{p^r,V}^{\otimes n} \quad \text{on} \quad V.$$

where g *denotes* $V \to V' := X'[1/p]$.

Remark. The morphism res^f is an isomorphism in the following cases (but *not*, otherwise):

- f is étale,
- $(m, \mathrm{ch}(k)) = 1$,
- $n \leq 0$,
- $\max\{\dim(X), \dim(X')\} < n$.

§3 Duality for arithmetic schemes

A : algebraic integer ring

Theorem. (Artin-Verdier duality)

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}_c^3(\mathrm{Spec}(A), \mathbb{G}_{\mathrm{m}}) \simeq \mathbb{Q}/\mathbb{Z}$.

(2) *For a constructible sheaf* $\mathcal{F}$ *on* $\mathrm{Spec}(A)_{\text{ét}}$, *the pairing*

$$\mathrm{H}_c^i(\mathrm{Spec}(A), \mathcal{F}) \times \mathrm{Ext}^{3-i}_{\mathrm{Spec}(A)}(\mathcal{F}, \mathbb{G}_{\mathrm{m}}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is a non-degenerate pairing of finite abelian groups, for $^{\forall}i$.

Remark.

- This duality is deduced from the Tate duality theorems for global fields and local fields.
- Generalized to '$\mathbb{Z}$-constructible' sheaves. (Deninger)
- Generalized to two-dimensional arithmetic schemes, replacing $\mathbb{G}_{\mathrm{m}}$ with a modified version of Lichtenbaum complex $\mathbb{Z}(2)$. (Spieß)

Corollary. (Duality outside of characteristic p)

Let V be an integral regular scheme which is flat separated of finite type over $\mathrm{Spec}(A[1/p])$. *Put* $d := \dim(V)$. *Then*:

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}_c^{2d+1}(V, \mu_{p^r}^{\otimes d}) \simeq \mathbb{Z}/p^r\mathbb{Z}$.

(2) *For a constructible $\mathbb{Z}/p^r\mathbb{Z}$-sheaf $\mathcal{F}$ on $V_{\text{ét}}$, the pairing*

$$\mathrm{H}_c^i(V, \mathcal{F}) \times \mathrm{Ext}^{2d+1-i}_{V,\mathbb{Z}/p^r\mathbb{Z}}(\mathcal{F}, \mu_{p^r}^{\otimes d}) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules, for $^{\forall}i$.

Proof. • The case $d = 1$: By the Kummer sequence

$$\mu_{p^r} \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mu_{p^r}[1],$$

reduced to the Artin-Verdier duality.

• The general case: $f : V \to \mathrm{Spec}(A[1/p])$ structure map.

We have a canonical isomorphism:

$$\mathrm{tr}^f : \mu_{p^r}^{\otimes d}[2(d-1)] \xrightarrow{\simeq} Rf^! \mu_{p^r} \quad \text{in } D^+(V_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$$

(Poincaré duality + absolute purity (Thomason-Gabber)).

Hence by Deligne's duality

$$Rf_* R\mathcal{H}om_{V,\mathbb{Z}/p^r\mathbb{Z}}(\mathcal{F}, Rf^! \mu_{p^r}) = R\mathcal{H}om_{\mathrm{Spec}(A[1/p]),\mathbb{Z}/p^r\mathbb{Z}}(Rf_! \mathcal{F}, \mu_{p^r}),$$

reduced to the 1-dimensional case. □

A remains to be global. Let $X \to \mathrm{Spec}(A)$ be as in §2.

Suppose X to be separated, and put $d := \dim(X)$.

Theorem III. (Jannsen - Saito - S)

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}_c^{2d+1}(X, \mathfrak{T}_r(d)_X) \simeq \mathbb{Z}/p^r\mathbb{Z}$.

(2) *For a constructible* $\mathbb{Z}/p^r\mathbb{Z}$*-sheaf* $\mathcal{F}$ *on* $X_{\text{ét}}$*, the pairing*

$$\mathrm{H}_c^i(X, \mathcal{F}) \times \mathrm{Ext}^{2d+1-i}_{X, \mathbb{Z}/p^r\mathbb{Z}}(\mathcal{F}, \mathfrak{T}_r(d)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$*-modules, for* $^{\forall}i$.

Proof. • The case $d = 1$: By the Kummer sequence

$$\mathfrak{T}_r(1)_X \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathfrak{T}_r(1)_X[1],$$

reduced to the Artin-Verdier duality.

• The general case: Glue the relative trace map for $X[1/p] \to \mathrm{Spec}(A[1/p])$ with those of fibers of characteristic p, and prove the isomorphism in $D^+(X_{\text{ét}}, \mathbb{Z}/p^r\mathbb{Z})$:

$$\mathrm{tr}^f : \mathfrak{T}_r(d)_X[2(d-1)] \xrightarrow{\simeq} Rf^!\mathfrak{T}_r(1)_{\mathrm{Spec}(A)}.$$

Then by the same argument as the previous corollary, reduced to the case $X = \mathrm{Spec}(A)$. □

Theorem IV. (S)

Suppose that X is proper over $\mathrm{Spec}(A)$*. Then for any integers n and i, the pairing (coming from Theorems* II *and* III (1))

$$\mathrm{H}^i_c(X, \mathfrak{T}_r(n)_X) \times \mathrm{H}^{2d+1-i}_{\text{ét}}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules.

Proof. $V := X[1/p]$, $Y :=$ union of fibers of characteristic p.

The case $n < 0$: the previous corollary.

The case $n \geq 0$: By the exact sequences

$$\cdots \to \mathrm{H}^i_c(X, j_!\mu_{p^r}^{\otimes n}) \to \mathrm{H}^i_c(X, \mathfrak{T}_r(n)_X) \to \mathrm{H}^i_{\text{ét}}(Y, \mathfrak{T}_r(n)_X|_Y) \to \cdots,$$

$$\cdots \to \mathrm{H}^{i'}_{Y,\text{ét}}(X, \mathfrak{T}_r(d-n)_X) \to \mathrm{H}^{i'}_{\text{ét}}(X, \mathfrak{T}_r(n)_X) \to \mathrm{H}^{i'}_{\text{ét}}(V, \mu_{p^r}^{\otimes d-n}) \to \cdots$$

with $i' := 2d + 1 - i$ and by the previous corollary, reduced to the following theorem:

Theorem V. (S)

Suppose that A is local and that X is proper over $\mathrm{Spec}(A)$.

(1) $^{\exists}$ *canonical trace map* $\mathrm{H}^{2d+1}_{Y,\text{ét}}(X, \mathfrak{T}_r(d)_X) \simeq \mathbb{Z}/p^r\mathbb{Z}$.

(2) *For any integers $n \geq 0$ and i, the pairing*

$$\mathrm{H}^i_{\text{ét}}(X, \mathfrak{T}_r(n)_X) \times \mathrm{H}^{2d+1-i}_{Y,\text{ét}}(X, \mathfrak{T}_r(d-n)_X) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$$

is a non-degenerate pairing of finite $\mathbb{Z}/p^r\mathbb{Z}$-modules.

Proof. (smooth case, for simplicity)

The case $n > d$: Tate duality for $K := \mathrm{Frac}(A)$ and

Poincaré duality for $V_{\overline{K}} = X \otimes_A \overline{K}$

The case $n = 0$: Milne duality for Y

The case $0 < n \leq d$: Reduced to the case $r = 1$ and $\zeta_p \in A$.

Let $\iota : Y \to X$ be the canonical map.

By the Bloch-Kato theorem on $R^q j_* \mu_p^{\otimes q}$, the cohomology sheaves of $\mathbb{Z}/p\mathbb{Z}(n)$ and $R\iota^! \mathbb{Z}/p\mathbb{Z}(d-n)$ are extensions of

$$\Omega^*_Y, \qquad d\Omega^*_Y, \qquad \Omega^*_{Y,\log}.$$

A key step is to show that the induced pairing on those pieces are the same as the wedge product of differential forms up to a non-zero constant (explicit formula). $\square$

§4 Consequences of duality results

Let p, A and X be as in Theorems IV or V. Put $d := \dim(X)$.

Corollary.

Suppose that A is local. Put $V := X[1/p] = X \otimes_A \mathrm{Frac}(A)$.

Define the 'unramified part' $\mathrm{H}^i_{\mathrm{ur}}(V, \mu_{p^r}^{\otimes n})$ as

$$\mathrm{Ker}\left(\mathrm{H}^i_{\text{ét}}(V, \mu_{p^r}^{\otimes n}) \longrightarrow \mathrm{H}^{i+1}_{Y,\text{ét}}(X, \mathfrak{T}_r(n)_X)\right).$$

Then under the non-degenerate pairing

$$\mathrm{H}^i_{\text{ét}}(V, \mu_{p^r}^{\otimes n}) \times \mathrm{H}^{2d-i}_{\text{ét}}(V, \mu_{p^r}^{\otimes d-n}) \longrightarrow \mathbb{Z}/p^r\mathbb{Z},$$

the subgroups $\mathrm{H}^i_{\mathrm{ur}}(V, \mu_{p^r}^{\otimes n})$ and $\mathrm{H}^{2d-i}_{\mathrm{ur}}(V, \mu_{p^r}^{\otimes d-n})$ are exact annihilators of each other.

Corollary. (Lichtenbaum)

Suppose that A is local and $d = 2$. (i.e., V is a curve.)

Then $\exists$ canonical non-degenerate pairing of finite groups:

$$\mathrm{Pic}(V)/p^r \times {}_{p^r}\mathrm{Br}(V) \longrightarrow \mathbb{Z}/p^r\mathbb{Z}.$$

Proof. Follows from the previous corollary and the equality

$$\mathrm{Pic}(V)/p^r = \mathrm{H}^2_{\mathrm{ur}}(V, \mu_{p^r}) \; (\subset \mathrm{H}^2_{\text{ét}}(V, \mu_{p^r}) \,). \qquad \square$$

Corollary. (Tate/Cassels/Saito)

Suppose that A is global, $d = 2$ and $p \geq 3$.

(i.e., X is an arithmetic surface.)

Define $\mathrm{Br}(X)_{p\text{-cotor}} := \mathrm{Br}(X)\{p\}/(\mathrm{Br}(X)\{p\})_{\mathrm{Div}}$ *(finite group).*

Then $^\exists$ *canonical non-degenerate alternating pairing*

$$\langle\ ,\ \rangle : \mathrm{Br}(X)_{p\text{-cotor}} \times \mathrm{Br}(X)_{p\text{-cotor}} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

In particular, the order of $\mathrm{Br}(X)_{p\text{-cotor}}$ *is a square number.*

Proof. Decompose the non-degenerate pairing (Theorem IV)

$$\begin{array}{ccccc} \mathrm{H}^2_{\text{ét}}(X, \mathfrak{T}_{\mathbb{Q}_p/\mathbb{Z}_p}(1)) & \times & \mathrm{H}^3_{\text{ét}}(X, \mathfrak{T}_{\mathbb{Z}_p}(1)) & \longrightarrow & \mathrm{H}^5_{\text{ét}}(X, \mathfrak{T}_{\mathbb{Q}_p/\mathbb{Z}_p}(2)) \\ & & & & \downarrow \simeq \\ & & & & \mathbb{Q}_p/\mathbb{Z}_p \end{array}$$

and compute signs of cup products □

§5 Euler-Poincaré characteristics

Let A, X and p be as in Theorem IV.

Problem.

∘ *For fixed i and n, is $\dim_{\mathbb{Q}_p} \mathrm{H}^i_{\text{ét}}(X, \mathfrak{T}_{\mathbb{Q}_p}(n)_X)$ independent of p?*

Concerning this problem, we have the following weaker result:

Theorem VI. *Put*

$$\begin{aligned} \chi(X, \mathfrak{T}_{\mathbb{Q}_p}(n)) &:= \sum_{i=0}^{\infty} (-1)^i \cdot \dim_{\mathbb{Q}_p} \mathrm{H}^i_{\text{ét}}(X, \mathfrak{T}_{\mathbb{Q}_p}(n)_X), \\ \chi_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) &:= \sum_{i=0}^{\infty} (-1)^i \cdot \dim_{\mathbb{R}} \mathrm{H}^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)), \end{aligned}$$

where $\mathrm{H}^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ denotes the real Deligne cohomology of $X_{\mathbb{C}/\mathbb{Z}} := X \otimes_{\mathbb{Z}} \mathbb{C}$:

$$\mathrm{H}^*_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) := \mathrm{H}^*_{\text{an}}(X_{\mathbb{C}/\mathbb{Z}}(\mathbb{C})^{\text{an}}, \mathbb{R}(n)_{\mathcal{D}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}.$$

Then we have

$$\chi(X, \mathfrak{T}_{\mathbb{Q}_p}(n)) = \chi_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)).$$

In particular, $\chi(X, \mathfrak{T}_{\mathbb{Q}_p}(n))$ is independent of p for which p-adic étale Tate twists are defined.