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Surface Bundles over S^1 Which Are 2-fold Branched Cyclic Coverings of S^3

Dedicated to the memory of Shouro Kasahara

By Makoto SAKUMA

Let F_g be a closed orientable surface of genus g. Then $F_g \times S^1$ $(g \geq 1)$ is not a 2-fold branched cyclic covering of S^3 (Fox [3], Hirsh-Neumann [5] and Montesinos [8]). Recently, M. Ochiai and M. Takahashi [10] showed that, for any positive integer g, there is an F_g-bundle over S^1 (F_g-bundle, in brief) which is a 2-fold branched cyclic covering of S^3. In fact, they showed that there is an F_g-bundle of Heegaard genus 2 for any positive integer g, and they classified torus bundles of Heegaard genus 2 (cf. [6]).

In this paper, we consider the problem of which surface bundle is a 2-fold branched cyclic covering of S^3. Let ϕ be an orientation preserving homeomorphism of F_g. Let M_ϕ be the space obtained from $F_g \times I$ by identifying $(x,0) \in F_g \times 0$ with $(\phi(x),1) \in F_g \times 1$. Then M_ϕ is an orientable F_g-bundle over S^1, and every such bundle is so obtained. The first integral homology group $H_1(M_\phi)$ is isomorphic to $Z \oplus Coker(\phi_* - 1)$, where ϕ_* is the automorphism of $H_1(F_g)$ induced by ϕ. Since $H_1(F_g) \cong Z^{2g}$, there are unique non-negative integers n_i $(1 \leq n_i \leq 2g)$, such that $Coker(\phi_* - 1) \cong \oplus_{i=1}^{2g} Z_{n_i}$ (where Z_0 denotes Z), and n_i divides n_{i+1} $(1 \leq i \leq 2g - 1)$.

We shall prove the following:

Theorem 1. *If* M_ϕ *is a 2-fold branched cyclic covering of a homology sphere, then* $n_g = 1$ *or* 2. *Coversely, let* n_i $(1 \le i \le 2g)$ *be non-negative integers, such that* n_i *divides* n_{i+1} $(1 \le i \le 2g-1)$, *and* $n_g = 1$ *or* 2. *Then there is an* F_g*-bundle* M, *such that* $H_1(M) \cong Z \oplus \{\oplus_{i=1}^{2g} Z_{n_i}\}$ *and* M *is a 2-fold branched cyclic covering of* S^3.

For torus bundles, we can list up all such bundles. In fact, we shall prove the following:

Theorem 2. *For a torus bundle* M, *the following three conditions are equivalent*:

(1) M *is a 2-fold branched cyclic covering of a homology sphere.*

(2) M *is a 2-fold branched cyclic covering of* S^3.

(3) M *is homeomorphic to* $M_{\alpha,\beta}$ *for some pair of integers* (α,β), *where* $M_{\alpha,\beta}$ *is the torus bundle whose monodromy is presented by the matrix* $A_{\alpha,\beta} = \begin{pmatrix} -1 & -\alpha \\ \beta & \alpha\beta - 1 \end{pmatrix}$.

$M_{\alpha,\beta}$ is the 2-fold branched cyclic covering of S^3, branched along the link $K(\alpha,\beta)$ as illustrated in Fig. 1.

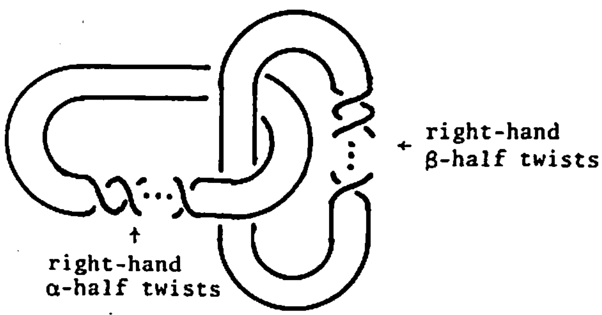

Fig. 1

Remark. (1) The link $K(\alpha,\beta)$ is equivalent to the link $K(\alpha',\beta')$, iff (α',β') is equal to $\pm(\alpha,\beta)$ or $\pm(\beta,\alpha)$. Thus, for any $K(\alpha,\beta)$, there is a unique pair of integers (α_0,β_0), such that (i) $1 \le \alpha_0 \le |\beta_0|$ or $0 = \alpha_0 \le \beta_0$ and (ii) $K(\alpha,\beta)$ is equivalent to $K(\alpha_0,\beta_0)$.

(2) $K(\alpha,\beta)$ is a 3-bridge link, iff α or β is equal to ± 1.

Furthermore, a slight generalization of Tollefson's theorem in [12] on involutions of surface bundles enables us to prove the following:

Theorem 3. *Let L be a link in S^3. Then the 2-fold branched cyclic covering of S^3 branched along L is a torus bundle, iff L is equivalent to the link $K(\alpha,\beta)$ for some pair of integers (α,β).*

Corollary. (Theorem 3 of [10]) *An orientable torus bundle M has Heegaard genus 2, iff M is homeomorphic to $M_{1,\beta}$ for some integer β.*

Remark. (1) The integer β is uniquely determined by M, and genus 2 Heegaard splitting of $M_{1,\beta}$ is unique, from Theorem 4 below (cf. [1]).

(2) Since $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} A_{1,\beta} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \beta - 2 & -1 \\ 1 & 0 \end{pmatrix}$, $M_{1,\beta}$ is homeomorphic to the torus bundle $M(\beta - 2,-1)$ defined in [10].

In the last section, we will give a practical method determining whether a given torus bundle is a 2-fold branched cyclic covering of S^3 or not (Theorem 5). To do this, we use a result of [7] or [11], and we will give complete invariants of the homeomorphism types of torus bundles (Lemmas 7 and 8). In particular, we will have the following:

Theorem 4. *Let (α_i,β_i) be a pair of integers such that $1 \le \alpha_i \le |\beta_i|$ or $0 = \alpha_i \le \beta_i$, for each $i = 1,2$. Then M_{α_1,β_1} is homeomorphic to M_{α_2,β_2}, iff*

(1) $(\alpha_1,\beta_1)=(\alpha_2,\beta_2)$ *or* (2) $(\alpha_i,\beta_i)=(1,6)$ *or* (2,3) *for each* $i=1,2$. *In other words, the branch line is unique with the exception of* $M_{1,6}\cong M_{2,3}$.

Finally, we will give a table of torus bundles M with $|TorH_1(M)|\le 20$. In particular, we will have the following:

Theorem 6. *Let* M *be a torus bundle, such that* $H_1(M)$ *is isomorphic to* $Z\oplus Z_n$ $(0\le n\le 11$, *or* $n=14$, 16 *or* 19) *or* $Z\oplus Z_2\oplus Z_{2n}$ $(0\le n\le 4)$. *Then* M *is a 2-fold branched cyclic covering of* S^3.

In this paper, we assume that every orientable surface F_g is endowed with a *longitude-meridian system* (*l-m system*, in brief) $\{l_i,m_i \mid 1\le i\le g\}$. For a self-homeomorphism ϕ of F_g, let A_ϕ be the matrix representing the automorphism ϕ_* of $H_1(F_g)$ induced by ϕ, with respect to the *l-m system*. Then A_ϕ is an element of $Sp(2g,Z)=\{A \mid AJA^t=\pm J\}$, where $J=\overset{g}{\oplus}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. The correspondence $\phi\to A_\phi$ induces an anti-homomorphism from $H(F_g)$, the homeotopy group of F_g, onto $Sp(2g,Z)$. ϕ is orientation preserving (resp. reversing), iff A_ϕ is an element of $Sp^+(2g,Z)=\{A \mid AJA^t=J\}$ (resp. $Sp^-(2g,Z)=\{A \mid AJA^t=-J\}$). If $g=1$, the anti-homomorphism is one to one, and $Sp(2,Z)=GL(2,Z)$, $Sp^+(2,Z)=SL(2,Z)$, and $Sp^-(2,Z)=SL^-(2,Z)=\{A \mid detA=-1\}$. For a matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of $GL(2,Z)$, let ϕ_A be the self-homeomorphism of $S^1\times S^1$ defined by $\phi_A(z_1,z_2)=(z_1^a\cdot z_2^c,\ z_1^b\cdot z_2^d)$, where S^1 is identified with the unit sphere in the complex plane. If the *l-m system* of $S^1\times S^1$ is given by $l=S^1\times 1$ and $m=1\times S^1$, then $A_{\phi_A}=A$. The torus bundle M_{ϕ_A} is denoted by M_A.

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1. Proof of Theorem 1

Lemma 1. *Let M be a 2-fold branched cyclic covering of a 3-manifold N, and let h be the covering transformation. Then the following hold:*

(1) *If N is a Z_p-homology sphere, for some non-negative integer p, then $1+h_* = 0 : H_i(M;Z_p) \to H_i(M;Z_p)$ $(i = 1,2)$.*

(2) *If $1+h_* = 0 : H_1(M) \to H_1(M)$, then the transfer $\tau : H_1(N) \to H_1(M)$ is a zero-map, and $2H_1(N) = 0$.*

Proof. This follows from a standard argument using transfer (see [2]) and the fact that the homomorphism $H_1(M) \to H_1(N)$ induced by the covering projection is an onto map.

Lemma 2. *Let M be a 2-fold branched cyclic covering of a Z_p-homology 3-sphere. Then, for any two elements x and y of $H_2(M;Z_p)$, $2int(x,y) = 0 \in H_1(M;Z_p)$, where int denotes the intersection pairing.*

Proof. Let h_* be the automorphism of $H_*(M;Z_p)$ induced by the covering transformation. Then we have the following, which proves Lemma 2.

$$-int(x,y) = h_*(int(x,y)) = int(h_*(x),h_*(y)) = int(-x,-y) = int(x,y).$$

Now we prove the first half of Theorem 1. Assume that M_ϕ is a 2-fold branched cyclic covering of a homology sphere and $n_g \geq 3$, where n_g is the integer defined in the introduction. Then there is an odd prime p dividing n_g. Let ϕ_* be the automorphism of $H_1(F_g;Z_p)$ induced by ϕ. Then $dim_{Z_p} Coker(\phi_*-1) \geq g+1$; so $dim_{Z_p} Im(\phi_*-1) \leq g-1$, and $dim_{Z_p} Ker(\phi_*-1) \geq g+1$. Let η be the natural map $H_1(F_g;Z_p) \to Coker(\phi_*-1) \subset H_1(M_\phi;Z_p)$. Then $dim_{Z_p} \eta(Ker(\phi_*-1)) \geq dim_{Z_p} Ker(\phi_*-1) - dim_{Z_p} Im(\phi_*-1) \geq (g+1)-(g-1) \geq 2$. Hence there is a 1-cycle z in F_g such that $[z] \in Ker(\phi_* - 1)$ and $(p-1)\eta([z]) \neq 0 \in H_1(M_\phi;Z_p)$, where $[z]$ is the homology class of $H_1(F_g;Z_p)$ represented by z. Since $\phi_*([z]) = [z]$, there is a 2-chain c such that $\partial c \equiv \phi_*([z]) - [z]$ mod p. Let $\hat{z}$ be the 2-chain of M_ϕ represented by $z \times I + c \times 0 \subset F_g \times I \,/\, (x,0) \sim (\phi(x),1) \cong M_\phi$. Then $\hat{z}$ is a mod p 2-cycle. Let $[\hat{z}]$ be the homology class of $H_2(M_\phi;Z_p)$ represented by $\hat{z}$, and let $[F_g]$ be the homology class of $H_2(M_\phi;Z_p)$ represented by $F_g \times (1/2)$. Then it can be seen that $int([\hat{z}],[F_g]) = \eta([z])$. Since $(p-1)\eta([z]) \neq 0$, this contradicts Lemma 2; so n_g is equal to 1 or 2.

Next we prove the later half of Theorem 1. Let V be an oriented handle body of genus g, with a fixed *l-m system* $\{l_i, m_i \mid 1 \leq i,j \leq g\}$. For a g-tuple of integers $(\alpha_1, \cdots, \alpha_g)$, let us consider a surface $S(\alpha_1, \cdots, \alpha_g)$ as shown in Fig. 2, consisting of one disk and g bands, where α_i denotes the number of half twists of the i-th band.

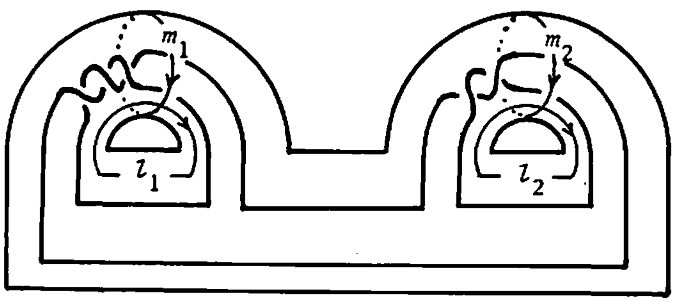

Fig. 2 $S(3,-2)$

Consider two such pairs of manifolds $(V_1, S_1(\alpha_1, \cdots, \alpha_g))$ and $(V_2, S_2(\beta_1, \cdots, \beta_g))$. Let $f : \partial V_1 \to \partial V_2$ be an orientation reversing homeomorphism. Then we obtain an oriented closed 3-manifold $N = V_1 \cup_f V_2$ and a surface $S_1 \cup S_2$ embedded in N. Let L be the link in N formed by the boundary of $S_1 \cup S_2$.

Consider the homomorphism $\psi:\pi_1(N-L)\to Z_2$, defined by $\psi(x) = int([x],[S_1]+[S_2])$ for each element x of $\pi_1(N-L)$, where $[x]$ denotes the homology class of $H_1(N-L;Z_2)$ represented by x, and $[S_1]$ (resp. $[S_2]$) denotes the homology class of $H_2(N,L;Z_2)$ represented by S_1 (resp. S_2). Then ψ sends the meridian of each component to the genetator of Z_2. Let M be the 2-fold branched cyclic covering of N branched along L corresponding to $Ker(\psi)$. Then we have the following:

Lemma 3. *M is an F_g-bundle with monodromy ϕ, such that $A_\phi = A_f P_2 A_f^{-1} P_1$, where A_f is the matrix representing the isomorphism $f_*: H_1(\partial V_1)\to H_1(\partial V_2)$, with respect to the given l-m systems of V_1 and V_2, $P_1 = \oplus_{i=1}^{g}\begin{pmatrix}1 & \alpha_i\\ 0 & -1\end{pmatrix}$ and $P_2 = \oplus_{i=1}^{g}\begin{pmatrix}1 & \beta_i\\ 0 & -1\end{pmatrix}$.*

Proof. Let $\hat{V}_i$ be the manifold obtained by cutting open V_i along the interior of S_i for each $i = 1,2$. Then $\hat{V}_i$ $(i = 1,2)$ is isomorphic to $F_g\times I$. Let C_i^- be the component of $\partial\hat{V}_i$ which contains ∂S_i, and let C_i^+ be the component of $\partial\hat{V}_i$ corresponding to ∂V_i for each $i = 1,2$. Take two copies $\hat{V}_i$ and $\hat{V}_i'$ of $\hat{V}_i$ for each $i = 1,2$. Then M is obtained by glueing $\hat{V}_1$, $\hat{V}_1'$, $\hat{V}_2$, and $\hat{V}_2'$, according to the following scheme:

$$\begin{array}{ccccccccccc} C_1^- & \subset & \hat{V}_1 & \supset & C_1^+ & \xrightarrow{f} & C_2^+ & \subset & \hat{V}_2 & \supset & C_2^- \\ \downarrow{\scriptstyle\gamma_1} & & & & & & & & & & \downarrow{\scriptstyle\gamma_2} \\ C_1^{-\prime} & \subset & \hat{V}_1' & \supset & C_1^{+\prime} & \xrightarrow{f} & C_2^{+\prime} & \subset & \hat{V}_2' & \supset & C_2^{-\prime} \end{array},$$

where γ_1 (resp. γ_2) denotes the involution of C_1^- (resp. C_2^-) such that $Fix(\gamma_1) = \partial S_1$ (resp. $Fix(\gamma_2) = \partial S_2$). Hence M is an F_g-bundle with monodromy $\phi = \gamma_1^{-1}\circ f^{-1}\circ\gamma_2\circ f$. It can be seen that, γ_{1*} (resp. γ_{2*}) is represented by the matrix $\oplus_{i=1}^{g}\begin{pmatrix}1 & \alpha_i\\ 0 & -1\end{pmatrix}$ (resp. $\oplus_{i=1}^{g}\begin{pmatrix}1 & \beta_i\\ 0 & -1\end{pmatrix}$), with respect to the *l-m system* of C_1^- (resp. C_2^-) induced by that of V_1 (resp. V_2). This completes the proof of Lemma 3.

In the above lemma, choose f to be the homeomorphism corresponding to the standard Heegaard splitting of S^3 of genus g. Then we have a link in S^3, whose 2-fold branched cyclic covering is an F_g-bundle M_ϕ such that $A_\phi = \oplus_{i=1}^{g}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & \beta_i\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & \alpha_i\\ 0 & -1\end{pmatrix} = \oplus_{i=1}^{g}\begin{pmatrix}-1 & -\alpha_i\\ \beta_i & \alpha_i\beta_i - 1\end{pmatrix}$. $H_1(M_\phi)$ is isomorphic to $Z\oplus\{\oplus_{i=1}^{g}G_i\}$, where G_i is the abelian group presented by the matrix $\begin{pmatrix}-2 & -\alpha_i\\ \beta_i & \alpha_i\beta_i - 2\end{pmatrix}$. Hence $H_1(M_\phi)\cong Z\oplus\{\oplus_{i=1}^{g}\{Z_{g_i}\oplus Z_{(|\alpha_i\beta_i-4|/g_i)}\}\}$, where $g_i = g.c.d.\{\alpha_i,\beta_i,2\}$. From this, we can prove the later half of Theorem 1.

2. Proof of Theorem 2

For an F_g-bundle M_ϕ, let $\tilde{M}_\phi$ be the infinite cyclic covering of M_ϕ, corresponding to $Ker(p_* : \pi_1(M_\phi)\to\pi_1(S^1))$, where $p : M_\phi\to S^1$ is the bundle projection. Then $\tilde{M}_\phi$ is homeomorphic to $F_g\times R^1$, and the homeomorphism $\hat{\phi}$: $F_g\times R^1\to F_g\times R^1$ defined by $\hat{\phi}(x,t) = (\phi(x),t+1)$ is a generator of the covering transformation group.

Lemma 4. *If* M_ϕ *is a 2-fold branched cyclic covering of a homology sphere, then there is a matrix* P *of* $Sp^-(2g,Z)$ *which satisfies the following conditions*:

(1) $P^2 = I$ *and* $A_\phi P = PA_\phi^{-1}$,

(2) $[P]+I = 0 : Coker(A_\phi - I)\to Coker(A_\phi - I)$, *where* $[P]$ *denotes the homomorphism induced by* P. *(Note that the condition* (1) *assures the existence of such a homomorphism.)*

Proof. Assume that M_ϕ is a 2-fold branched cyclic covering of a homology sphere, and let h be the covering transformation. Then, from Lemma 1 (1), there is a lift $\tilde{h}$ of h, such that $\tilde{h}\circ\hat{\phi} = \hat{\phi}^{-1}\circ\tilde{h}$ and $\tilde{h}_*([F_g])=-[F_g]$ in $H_2(\tilde{M}_\phi)$, where $[F_g]$ is the homology class represented by $F_g\times 0 \subset F_g\times R^1 = \tilde{M}_\phi$. Hence the matrix P representing $\tilde{h}_* : H_1(\tilde{M}_\phi) \to H_1(\tilde{M}_\phi)$ is an element of $Sp^-(2g,Z)$, and it satisfies the later half of the condition (1). Since $Fix(h) \neq \emptyset$, we can choose $\tilde{h}$ so that $Fix(\tilde{h}) \neq \emptyset$. Then $\tilde{h}^2 = 1$, since $\tilde{h}^2$ is a lift of $h^2 = 1$. So, we have $P^2 = I$. The condition (2) follows from the fact that $H_1(M_\phi) \cong Z \oplus Coker(A_\phi - I)$ and Lemma 1 (1).

To establish Theorem 2, we have only to prove that (1) implies (3). Let ϕ be a self-homeomorphism of a torus T^2, and assume that the torus bundle M_ϕ is a 2-fold branched cyclic covering of a homology sphere. We may assume that $\phi = \phi_{A_\phi}$. Let P be the matrix of $SL^-(2,Z)$ which satisfies the conditions of Lemma 4, and let $\gamma = \phi_P$. By the condition (1), we can define an involution h of $M_\phi = T^2\times R^1/(x,t)\sim(\phi(x),1+t)$ by the equation $h([x,t]) = [\gamma(x),1-t]$, where $[x,t]$ denotes the point of M_ϕ corresponding to the point (x,t) in $T^2\times R^1$. Then we have:

$$M_\phi/h \cong T^2\times[0,1/2]/\{(x,0)\sim(\phi^{-1}\circ\gamma(x),0),\ (x,1/2)\sim(\gamma(x),1/2)\} = V_1 \cup V_2,$$

where $V_1 = T^2\times[0,1/4]/(x,0)\sim(\phi^{-1}\circ\gamma(x),0)$, and

$$V_2 = T^2\times[1/4,1/2]/(x,1/2)\sim(\gamma(x),1/2).$$

Since γ and $\phi^{-1}\circ\gamma$ are orientation reversing involutions of T^2, A_γ and $A_{\phi^{-1}\circ\gamma}$ are conjugate to the matrix $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. From this, it can be seen that V_1 and V_2 are handlebodies of genus 1, and the links $Fix(\phi^{-1}\circ\gamma)\times 0 \subset V_1$ and $Fix(\gamma)\times(1/2) \subset V_2$ are equivalent to the links as illustrated in Fig. 2 in §1. Hence M_ϕ/h is a 3-manifold of Heegaard genus 1, and M_ϕ is a 2-fold branched branched cyclic covering of M_ϕ/h. It can be seen that $1+h_* = 0 : H_1(M_\phi) \to H_1(M_\phi)$, by the condition (2). Hence, from Lemma 1 (2), M_ϕ/h is S^3 or RP^3. If M_ϕ/h is S^3, the branch line is equivalent to the link $K(\alpha,\beta)$ as illustrated in Fig. 1 for some pair of integers (α,β), and we have the desired result. Thus we have only to prove that M_ϕ/h is not RP^3. From the fact that $1+h_* = 0$ and Lemma 1 (2) this is proved by the following Lemma.

Lemma 5. *Consider two links in solid tori* $(V_1, \partial S_1(\alpha))$ *and* $(V_2, \partial S_2(\beta))$ *defined in* §1, *and let* $f : \partial V_1 \to \partial V_2$ *be the homeomorphism which is represented by the matrix* $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ *with respect to the given* l-m *systems of* V_1 *and* V_2. *Let* M *be the 2-fold branched cyclic covering of* $V_1 \cup_f V_2 \cong RP^3$ *branched along the link* $\partial S_1(\alpha) \cup \partial S_2(\beta)$ *corresponding to* $Ker(\psi)$, *where* ψ *is the homomorphism definened in* §1. *Then the transfer* $\tau : H_1(RP^3) \to H_1(M)$ *is not a zero-map.*

Proof. From Lemma 3, M is a torus bundle whose monodromy is presented by the matrix $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1+2\beta & \alpha+\beta+2\alpha\beta \\ 4(1+\beta) & 1+4\alpha+2\beta+4\alpha\beta \end{pmatrix}$ with respect to the l-m *system* $\{l,m\}$ induced by that of V_1. Hence $H_1(M)$ is isomorphic to $Z \oplus \langle l,m \mid 2\beta l + (\alpha+\beta+2\alpha\beta)m,\ 4(1+\beta)l + 2(2\alpha+\beta+2\alpha\beta)m \rangle$. It is easy to see that $Im(\tau)$ is generated by $2l+\alpha m$. Assume that $Im(\tau) = 0$. Then there are integers x and y satisfying the following equations:

(1) $2\beta x + 4(1+\beta)y = 2$ (2) $(\alpha+\beta+2\alpha\beta)x + 2(2\alpha+\beta+2\alpha\beta)y = \alpha$.

By (1), we have $\beta \equiv x \equiv 1$ *mod*.2. By (2), we have $(\alpha+\beta)x \equiv \alpha$ *mod*.2. So $\alpha+1 \equiv \alpha$ *mod*.2; contradiction. Hence $Im(\tau) \neq 0$.

3. Proof of Theorem 3

J.L. Tollefson [12] proved the following:

Theorem. (Theorem 2 of [12]) *Let h be a P.L. involution on $M_\phi = F_g \times R^1/\hat{\phi}$, where $H_1(M_\phi;Q) \cong Q$. Then h is equivalent to an involution h' defined on $M_\zeta = F_g \times R^1/\hat{\zeta}$ by $h'([x,t]) = [\gamma(x),\lambda(t)]$, where γ is some involution on F_g, the map ζ is isotopic to ϕ, and $\lambda(t) = t$, $1-t$, or $t+(1/2)$.*

The condition $H_1(M_\phi;Q) \cong Q$ is used only in p.229 of [12], to prove that $h(F_g)$ is isotopic to F_g, where F_g is a fiber. This condition can be replaced with the condition that $h_*([F_g]) = \pm[F_g]$ in $H_2(M_\phi)$. In fact, it assures the existence of the lift of h to the infinite cyclic covering $\tilde{M}_\phi$ of M_ϕ, by which we can prove that $h(F_g)$ is isotopic to F_g from a similar argument to that of [12] in p.229.

From the above argument and Lemma 1 (1), we have the following:

Lemma 6. *Assume that M_ϕ is a 2-fold branched cyclic covering of a homology sphere. Then the covering transformation h is equivalent to an involution h' defined on $F_g \times R^1/\hat{\zeta}$ by $h'([x,t]) = [\gamma(x),1-t]$, where γ is some orientation reversing involution on F_g and the map ζ is isotopic to ϕ.*

Using Lemma 6, we can prove Theorem 3 by a similar argument to the proof of Theorem 2.

4. Invariants of torus bundles

Lemma 7. (1) *Let A be a matrix of $SL(2,Z)$ such that $H_1(M_A)$ is isomorphic to $Z \oplus Z \oplus Z_n$ for some non-negative integer n. Then A is* *conjugate** *to the matrix* $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

(2) *Let* A *and* A' *be matrices of* $SL(2,\mathbb{Z})$. *Then* M_A *is homeomorphic to* $M_{A'}$, *iff* A' *is conjugate to* A *or* A^{-1}.

Proof. (1) follows from the fact that $Ker(A-I)$ is non-trivial iff the first Betti number of M_A is greater than 1. (2) follows from (1) and the fact that two surface bundles, whose first Betti numbers are equal to 1, are homeomorphic, iff they are equivalent as fiber bundles (see [9]).

Thus the homeomorphism problem of torus bundles is reduced to the conjugacy problem for 2×2-matrices over $\mathbb{Z}$, which is solved as follows:

Lemma 8. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a matrix of* $SL(2,\mathbb{Z})$. *Then the following hold*:

(1) *The characteristic polynomial* $f_A(x)$ *of* A *is equal to* $x^2 - Tr(A)x + 1$, *and the discriminant* D_A *of* $f_A(x)$ *is equal to* $(Tr(A))^2 - 4$.

(2) *If* $Tr(A) = 2\varepsilon$ *for some* $\varepsilon = \pm 1$, *then there is a unique non-negative integer* n *such that* A *is conjugate to the matrix* $\begin{pmatrix} \varepsilon & 0 \\ n & \varepsilon \end{pmatrix}$.

(3) *If* $Tr(A) = -1$ (*resp.* 0, 1), A *is conjugate to the matrix* $A_{1,1}$ (*resp.* $A_{1,2}$, $A_{1,3}$).

(4) *If* $|Tr(A)| \geq 3$, $f_A(x)$ *is irreducible over* $\mathbb{Z}$, *and* D_A *is positive and non-square. Let* $\theta(A) = \{(d-a) + \sqrt{D_A}\}/2b$. *Then the conjugate class of* A *is completely determined by* $Tr(A)$ *and the equivalence class of the quadratic irrationality* $\theta(A)$. *Hence two matrices* A *and* A' *of* $Sl(2,\mathbb{Z})$, *such that* $Tr(A) = Tr(A') \neq 0, \pm 1, \pm 2$, *are conjugate, iff the purely cyclic parts of the infinite continued fractions representing* $\theta(A)$ *and* $\theta(A')$ *are equal up to cyclic permutations.*

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* This means "conjugate in $GL(2,\mathbb{Z})$".

Proof. (2). If $Tr(A) = 2\varepsilon$ for some $\varepsilon = \pm 1$, then $f_A(x) = (x-\varepsilon)^2$ and A has a real eigen value; so A is conjugate to the matrix $\begin{pmatrix} \varepsilon & 0 \\ n & \varepsilon \end{pmatrix}$ for some integer n. Since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ n & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \varepsilon & 0 \\ -n & \varepsilon \end{pmatrix}$, we may assume that n is non-negative. The first elementary ideal of the matrix $xI - \begin{pmatrix} \varepsilon & 0 \\ n & \varepsilon \end{pmatrix}$ is $<x-\varepsilon, n>$ and $Z[x]/<x-\varepsilon, n> \cong Z_n$. Hence the non-negative integer n is uniquely determined by the conjugate class of A.

(3) and (4). Assume that $Tr(A) \neq \pm 2$. Then $f_A(x)$ is irreducible over Z, and we can make use of the results of [7] or [11]. Let ξ be the first root of $f_A(x)$, that is, $\xi = \{Tr(A)+\sqrt{D_A}\}/2$. Then there are algebraic integers ω_1 and ω_2 of $Z[\xi]$, such that $A\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \xi\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$. [7] and [11] proved that the set $\{\omega_1, \omega_2\}$ forms a base of an ideal of the ring $Z[\xi]$, and the ideal class of the ideal $<\omega_1, \omega_2>$ is uniquely determined by the conjugate class of A. (3) follows from Lemma 9 below and the fact that the class numbers of the rings $Z[(\pm 1+\sqrt{3}i)/2]$ and $Z[i]$ are equal to 1. (4) is deduced from the following facts (i) ~ (iii): (i) two ideals $<\omega_1,\omega_2>$ and $<\omega_1',\omega_2'>$ are in the same ideal class, iff the quadratic irrationalities ω_2/ω_1 and ω_2'/ω_1' are equivalent, (ii) two quadratic irrationalities are equivalent, iff the purely cyclic parts of the infinite continued fractions representing the quadratic irrationalities are equal up to cyclic permutations (see [4]), (iii) $A\begin{pmatrix} b \\ \xi-a \end{pmatrix} = \xi\begin{pmatrix} b \\ \xi-a \end{pmatrix}$, and $(\xi-a)/b = \{(d-a)+\sqrt{D_A}\}/2b$.

For the matrix $A_{\alpha,\beta}$, we have the following:

Lemma 9. *Let* (α,β) *be a pair of integers, such that* $1 \leq \alpha \leq |\beta|$ *or* $0 = \alpha \leq \beta$. *Then the following hold:*

(1) $A_{\alpha,\beta}$ *is conjugate to* $A_{\alpha,\beta}^{-1}$.

(2) $Tr(A_{\alpha,\beta}) = -2$, *iff* $\alpha = 0$. $A_{0,\beta}$ *is equal to* $\begin{pmatrix} -1 & 0 \\ \beta & -1 \end{pmatrix}$.

(3) $Tr(A_{\alpha,\beta}) = 2$, *iff* $(\alpha,\beta) = (1,4)$ *or* $(2,2)$. $A_{1,4}$ *is conjugate to* $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, *and* $A_{2,2}$ *is conjugate to* $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

(4) $Tr(A_{\alpha,\beta}) = -1$ (*resp.* 0, 1), *iff* $(\alpha,\beta) = (1,1)$ (*resp.* $(1,2)$, $(1,3)$).

(5) *If* $|Tr(A_{\alpha,\beta})| \geq 3$, *the purely cyclic part of the infinite continued fraction representing* $\theta(A_{\alpha,\beta})$ *is given by the following formula up to cyclic permutations*:

$[\dot{\alpha}, |\dot{\beta}|]$ *if* $\beta < 0$,

$[\dot{1}, (\beta\dot{-}4)]$ *if* $\alpha = 1$ *and* $\beta \geq 5$,

$[\dot{2}, (\beta\dot{-}2)]$ *if* $\alpha = 2$ *and* $\beta \geq 3$,

$[\dot{1}, (\alpha-2), 1, (\beta\dot{-}2)]$ *if* $\alpha \geq 3$ *and* $\beta \geq 3$.

Proof. (1) follows from Lemma 4 (1). (2),(3), and (4) are trivial.

(5) By direct calculation, we can prove that $\theta(A_{\alpha,\beta})$ is represented by the following infinite continued fraction:

$[-1, (|\beta|+1), \dot{1}, |\dot{\beta}|]$ if $\alpha = 1$ and $\beta < 0$,

$[-1, 1, (\alpha-1), |\dot{\beta}|, \dot{\alpha}]$ if $\alpha \geq 2$ and $\beta < 0$,

$[(-\beta+1), (\beta-3), \dot{1}, (\beta\dot{-}4)]$ if $\alpha = 1$ and $\beta \geq 5$,

$[-\beta, 1, 1, (\beta\dot{-}2), \dot{2}]$ if $\alpha = 2$ and $\beta \geq 3$,

$[-\beta, (\alpha-1), \dot{1}, (\beta-2), 1, (\alpha\dot{-}2)]$ if $\alpha \geq 3$ and $\beta \geq 3$.

Now, Theorem 4 in the introduction and Theorem 5 below are immediate consequences of Lemmas 7, 8, and 9.

Theorem 5. *Let* A *be a matrix of* $SL(2,\mathbb{Z})$. *Then* M_A *is a 2-fold branched cyclic covering of* S^3, *iff one of the following conditions holds*:

(1) $-2 \leq Tr(A) \leq 1$,

(2) $Tr(A) = 2$ *and* $Coker(A - I) \cong \mathbb{Z}$ *or* $\mathbb{Z} \oplus \mathbb{Z}_2$

(3) $|Tr(A)| \geq 3$ *and there is a pair of integers* (α,β), *such that* $\alpha\beta = Tr(A) + 2$ *and the purely cyclic part of the infinite continued fraction representing* $\theta(A)$ *is equal to that representing* $\theta(A_{\alpha,\beta})$, *which is given by Lemma* 9 (5), *up to cyclic permutations.*

Example 1. Let $A = \begin{pmatrix} -5 & 3 \\ 8 & -5 \end{pmatrix}$. Then $H_1(M_A) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}$; so M_A satisfies the necessary condition given by Theorem 1. Nevertheless M_A is not a 2-fold branched cyclic covering of S^3. In fact, $Tr(A) = -10$ and $\theta(A) = \sqrt{96}/6 = [1,\dot{1},1,1,\dot{3}]$; so A does not satisfy the condition of Theorem 5.

Example 2. Let M be a torus bundle such that $dim_{\mathbb{Q}} H_1(M;\mathbb{Q}) \geq 2$. Then M is a 2-fold branched cyclic covering of S^3, iff $H_1(M)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

Now we will give a list of orientable torus bundles M, such that $dim_{\mathbb{Q}} H_1(M;\mathbb{Q}) = 1$ and $|TorH_1(M)| \leq 20$. To do this, we need the following lemma, which is proved by direct calculation.

Lemma 10. (1) *For a matrix* A *of* $SL(2,\mathbb{Z})$, *the following hold:*

(i) $dim_{\mathbb{Q}} H_1(M_A;\mathbb{Q}) \geq 2$, *iff* $Tr(A) = 2$,

(ii) *If* $Tr(A) \neq 2$, *then* $|TorH_1(M_A)| = |Tr(A) - 2|$ *and* D_A *is equal to* $|TorH_1(M_A)| \cdot \{|TorH_1(M_A)| \pm 4\}$.

(2) *Let* m *be an integer such that* $|m| \geq 3$, *and let* ξ *be the first root of a polynomial* $px^2 + qx + r$ *such that* $q^2 - 4pr = m^2 - 4$. *Define* $A(\xi)$ *to be the matrix* $\begin{pmatrix} (m+q)/2 & p \\ -r & (m-q)/2 \end{pmatrix}$. *Then* $A(\xi)$ *is an element of* $SL(2,\mathbb{Z})$, $Tr(A(\xi)) = m$, *and* $\theta(A(\xi)) = \xi$.

Thus to classify torus bundles M, such that $|TorH_1(M)| = n$, we have only to classify quadratic irrationalities which are roots of quadratic polynomials whose discriminants are equal to $n(n \pm 4)$.

In the following list, the torus bundle $M_{\alpha,\beta}$ is denoted by (α,β). A torus bundle M_A, which is not a 2-fold branched cyclic covering of S^3, is represented by A, and the purely cyclic part of the infinite continued fraction representing $\theta(A)$ is written on the right of A.

$H_1(M)$	*Torus bundles*
Z	(1,3), (1,5)
$Z \oplus Z_2$	(1,2), (1,6) $\cong$ (2,3)
$Z \oplus Z_3$	(1,1), (1,7)
$Z \oplus Z_4$	$(0,2\beta+1)$ where $\beta \in \mathbb{N} \cup \{0\}$, (1,8)
$Z \oplus Z_2 \oplus Z_2$	$(0,2\beta)$ where $\beta \in \mathbb{N} \cup \{0\}$, (2,4)
$Z \oplus Z_5$	(1,-1), (1,9), (3,3)
$Z \oplus Z_6$	(1,-2), (1,10), (2,5)
$Z \oplus Z_7$	(1,-3), (1,11)
$Z \oplus Z_8$	(1,-4), (1,12), (3,4)
$Z \oplus Z_2 \oplus Z_4$	(2,-2), (2,6)
$Z \oplus Z_9$	(1,-5), (1,13)
$Z \oplus Z_3 \oplus Z_3$	$\begin{pmatrix} -5 & 3 \\ 3 & -2 \end{pmatrix}$ ** $[\dot{1}]$, $\begin{pmatrix} 4 & 3 \\ 9 & 7 \end{pmatrix}$ ** $[\dot{3}]$
$Z \oplus Z_{10}$	(1,-6), (2,-3), (1,14), (2,7)
$Z \oplus Z_{11}$	(1,-7), (1,15), (3,5)
$Z \oplus Z_{12}$	(1,-8), (1,16), $\begin{pmatrix} -5 & 3 \\ 8 & -5 \end{pmatrix}$ * $[\dot{1},1,1,\dot{2}]$, $\begin{pmatrix} 7 & 3 \\ 16 & 7 \end{pmatrix}$ * $[\dot{3},\dot{4}]$
$Z \oplus Z_2 \oplus Z_6$	(2,-4), (2,8), (4,4)

$Z \oplus Z_{13}$	(1,-9), (3,-3), (1,17), $\begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}$ $[\dot{1},2,2,\dot{1}]$
$Z \oplus Z_{14}$	(1,-10), (2,-5), (1,18), (2,9), (3,6)
$Z \oplus Z_{15}$	(1,-11), (1,19), $\begin{pmatrix} -8 & 3 \\ 13 & -5 \end{pmatrix}$ ** $[\dot{1},1,1,\dot{3}]$, $\begin{pmatrix} 7 & 3 \\ 23 & 10 \end{pmatrix}$ ** $[\dot{3},\dot{5}]$
$Z \oplus Z_{16}$	(1,-12), (3,-4), (1,20), (4,5)
$Z \oplus Z_2 \oplus Z_8$	(2,-6), (2,10)
$Z \oplus Z_4 \oplus Z_4$	$\begin{pmatrix} -7 & 4 \\ 12 & -7 \end{pmatrix}$ * $[\dot{1},\dot{2}]$, $\begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix}$ * $[\dot{4}]$, $\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$ ** $[\dot{1}]$
$Z \oplus Z_{17}$	(1,-13), (1,21), (3,7), $\begin{pmatrix} -10 & 7 \\ 7 & -5 \end{pmatrix}$ $[\dot{1},2,2,\dot{1}]$
$Z \oplus Z_{18}$	(1,-14), (2,-7), (1,22), (2,11), $\begin{pmatrix} 7 & 9 \\ 10 & 13 \end{pmatrix}$ $[\dot{1},2,3,\dot{1}]$
$Z \oplus Z_3 \oplus Z_6$	$\begin{pmatrix} -8 & 3 \\ 21 & -8 \end{pmatrix}$ * $[\dot{1},1,4,\dot{1}]$, $\begin{pmatrix} 10 & 3 \\ 33 & 10 \end{pmatrix}$ * $[\dot{6},\dot{3}]$
$Z \oplus Z_{19}$	(1,-15), (3,-5), (1,23)
$Z \oplus Z_{20}$	(1,-16), (1,24), (3,8), $\begin{pmatrix} -9 & 5 \\ 16 & -9 \end{pmatrix}$ * $[\dot{1},2,1,\dot{3}]$
	$\begin{pmatrix} 11 & 8 \\ 15 & 11 \end{pmatrix}$ * $[\dot{2},1,2,\dot{2}]$, $\begin{pmatrix} 9 & 4 \\ 29 & 13 \end{pmatrix}$ ** $[\dot{4},\dot{5}]$
$Z \oplus Z_2 \oplus Z_{10}$	(2,-8), (4,-4), (2,12), (4,6), $\begin{pmatrix} -13 & 8 \\ 8 & -5 \end{pmatrix}$ ** $[\dot{1}]$

Remark. For a matrix A dotted by * (resp. **), the equation $AP = PA^{-1}$ holds, where $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$); so it can be seen that M_A is a 2-fold branched cyclic covering of a certain lens space which is not $S^2 \times S^1$, from a similar argument to the proof of Theorem 2. Furthermore we can see that M_A is a $Z_2 \oplus Z_2$-branched covering of S^3, from a similar argument to that of [5]. On the other hand, it can be seen that, there are no such equations for the matrices which are not dotted by * or **, and the corresponding torus bundles are not 2-fold branched cyclic coverings of the lens spaces other than $S^2 \times S^1$. In particular, for the matrix $A = \begin{pmatrix} 7 & 9 \\ 10 & 13 \end{pmatrix}$, $\theta(A) = [\dot{1},2,3,\dot{1}]$ and $\theta(A^{-1}) = [-1,4,\dot{2},1,1,\dot{3}]$; so A is not conjugate to A^{-1}.

Addendum 1.

From Lemma 3, it can be seen that every closed orientable 3-manifold has a surface bundle as a 2-fold branched cyclic covering. In other words, every closed orientable 3-manifold is a quotient space of a surface bundle by an involution.

Addendum 2.

F. Raymond and J.L. Tollefson asserted that a certain family of surface bundles $\{M_\Phi\}$ admit no nontrivial periodic maps in their paper "Closed 3-manifolds with no periodic maps" Trans. A.M.S. 221 (1976). Nevertheless, it appears not to be valid. Here we show that $\{M_\Phi\}$ are 2-fold branched cyclic coverings of S^3, and therefore they admit nontrivial involutions.

The surface homeomorphism Φ is defined by the notion of a twist map. For a simple closed curve c on F_g, let $t(c)$ be the twist map about c. We adopt the convention that $t(c)$ moves points on a direct line segment which is approaching c to the right. Now consider a closed orientable surface F_g $(g \geq 3)$ embedded in R^3 as illustrated in Fig. 3, and let $\{a_i, b_i\ (1 \leq i \leq g)\}$ denote the set of simple closed curves shown.

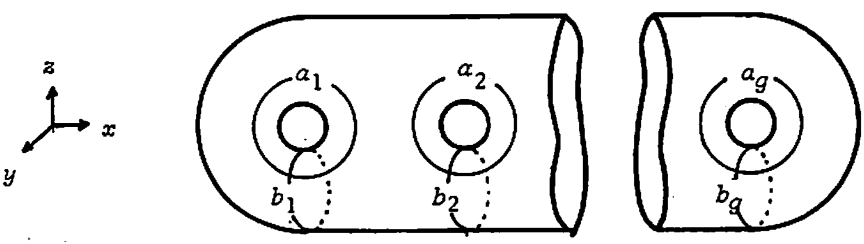
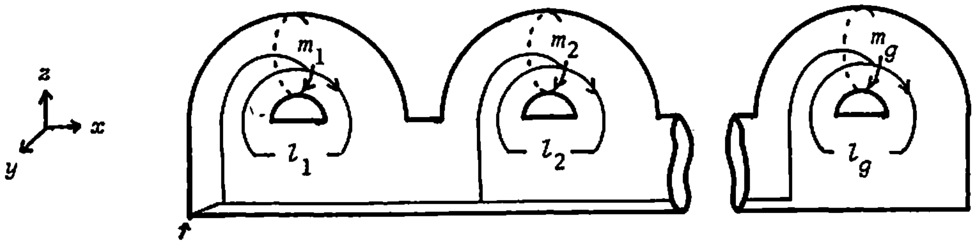


Fig. 3

Fix a set $\{n_1,\cdots,n_g\}$ of arbitrary but distinct positive integers, each greater than 2, and define Φ to be the homeomorphism of the surface obtained by setting $\Phi = \Pi_{i=1}^{g} t(a_i)\circ t(b_i)^{-n_i+1}$. To adjust these notations to the notations used in §1, we identify $(R^3,F_g,\{a_i,b_i\})$ in Fig. 3 with $(R^3,F_g,\{l_i,m_i\})$ in Fig. 4 by an orientation reversing homeomorphism of R^3.



the base point

Fig. 4

Then $t(a_i)$ (resp. $t(b_i)$) corresponds to $t(l_i)^{-1}$ (resp. $t(m_i)^{-1}$); so Φ corresponds to $\Phi' = \Pi_{i=1}^{g} t(l_i)^{-1}\circ t(m_i)^{n_i-1}$. Generators of the fundamental group $\pi_1(F_g)$ are taken as illustrated in Fig. 4, and we use the same symbol l_i (resp. m_i) to denote the element of $\pi_1(F_g)$ corresponding to the loop l_i (resp. m_i). Let us look at the automorphism of $\pi_1(F_g)$ induced by the twist maps $t(l_i)$ and $t(m_i)$. Each generator is fixed by $t(l_i)_\#$ except m_i which is mapped to $\bar{l}_i m_i$ and $t(m_i)_\#$ sends l_i to $m_i l_i$ and fixes the remaining generators. So we have $\Phi'_\#(l_i) = (l_i m_i)^{n_i-1} l_i$ and $\Phi'_\#(m_i) = l_i m_i$, and $\Phi'_* : H_1(F_g) \to H_1(F_g)$ is represented by the matrix $A_{\Phi'} = \oplus_{i=1}^{g}\begin{pmatrix} n_i & n_i-1 \\ 1 & 1 \end{pmatrix}$. Note that $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} n_i & n_i-1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = A_{n_i+3,1}$, where $A_{n_i+3,1}$ is the matrix given by Theorem 2 and $A_{n_i+3,1}$ is conjugate to

$A_{n_i+3,1}{}^{-1}$ by Lemma 4. So the matrix $\begin{pmatrix} n_i & n_i-1 \\ 1 & 1 \end{pmatrix}$ is conjugate to its inverse; this contradicts Lemma 1 of the preceding paper of Raymond and Tollefson.

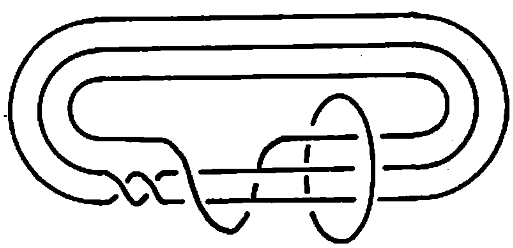
The above observation in cooperation with Lemma 3 provides a candidate for a link in S^3 which has M_Φ ($\cong M_{\Phi'}$) as the 2-fold branched cyclic covering. Consider two pairs of manifolds $(V_1, S_1(n_1+3,\cdots,n_g+3))$ and $(V_2, S_2(1,\cdots,1))$ defined in §1. Let $f: \partial V_1 \to \partial V_2$ be the homeomorphism corresponding to the standard Heegaard splitting of S^3. Let L be the link $\partial F_1 \cup \partial F_2$ in $V_1 \underset{f}{\cup} V_2 \cong S^3$. Then, by Lemma 3, the 2-fold branched cyclic covering M of S^3 branched along L is an F_g-bundle with monodromy $\phi = \gamma_1^{-1} \circ f^{-1} \circ \gamma_2 \circ f$, where γ_1 and γ_2 are homeomorphisms defined in the proof of Lemma 3. We claim that M is homeomorphic to $M_{\Phi'}$. To prove this, we show that $\phi_\#$ is conjugate to $\Phi'_\#$ in $Aut(\pi_1(F_g))$. It can be seen that $f_\#$, $\gamma_{1\#}$ and $\gamma_{2\#}$ are given by the formulas $f_\#(l_i) = W_i m_i \overline{W}_i$, $f_\#(m_i) = W_i l_i \overline{W}_i$, $\gamma_{1\#}(l_i) = W_i m_i^{n_i+2} l_i m_i \overline{W}_i$, $\gamma_{1\#}(m_i) = W_i \overline{m}_i \overline{W}_i$, $\gamma_{2\#}(l_i) = W_i l_i m_i \overline{W}_i$, and $\gamma_{2\#}(m_i) = W_i \overline{m}_i \overline{W}_i$, where $W_i = \Pi_{j=1}^{i-1} \overline{m}_j \overline{l}_j m_j l_j$. Furthermore the relations $f_\#^2 = \gamma_{1\#}^2 = \gamma_{2\#}^2 = Id$ and $f_\#(W_i) = \gamma_{1\#}(W_i) = \gamma_{2\#}(W_i) = \overline{W}_i$ hold. From this it can be seen that $\phi_\#(l_i) = \overline{m}_i \overline{l}_i \overline{m}_i^{(n_i+2)}$ and $\phi_\#(m_i) = m_i^{n_i+1} l_i m_i$. Now let $\psi = \Pi_{i=1}^{g} t(l_i)^{-1} \circ t(m_i)^{-2}$. Then, by a direct calculation, we have $\phi_\# = \psi_\#^{-1} \circ \Phi'_\# \circ \psi_\#$. Hence, by a well-known theorem of Nielsen, ϕ is isotopic to $\psi^{-1} \circ \Phi' \circ \psi$; so M is homeomorphic to M_Φ.

Addendum 3.

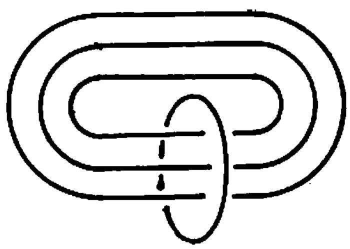
Here we give an affirmative answer to the following problem (Problem 25 of "Knot Theory, Proccedings, Plan-sur-Bex, Switzerland 1977, Lect. Notes in Math. 685, Springer-Verlag, 1978, p.311")

Problem. *Do there exist links in* S^3 *with the same comlement which are distinguished by the first Betti numbers of their 2-fold branched covers?*

Consider the link $K_{2,2}$ defined in the introduction. It is equivalent to the link as illustrated in Fig. 5 (a). So it can be seen that its complement is homeomorphic to the complement of the link K' as illustrated in Fig. 5 (b).



(a) $K_{2,2}$



(b) K'

Fig. 5

The 2-fold branched cyclic covering of $K_{2,2}$ is a torus bundle whose first Betti number is equal to 2. On the other hand, the 2-fold branched cyclic covering of K' is homeomorphic to $RP^3 \# RP^3 \# RP^3$; so its first Betti number is equal to 0. This gives an affirmative answer to the preceding problem.

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