SAKUMA, M
KOBE J. MATH.,
7 (1990) 167—190

THE GEOMETRIES OF SPHERICAL MONTESINOS LINKS

Dedicated to Professor Junzo Tao for his 60th birthday

By Makoto SAKUMA
(Received June 2, 1990)

A *Montesinos link* $L = L(b; (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r))$ with r branches is a link in S^3 as illustrated in Figure 0.1 (cf. [BiZ1, BnS2, BuZ, M1,2, Z]).

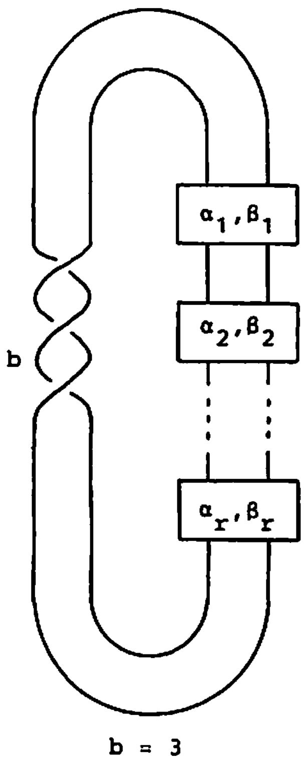


Fig. 0.1

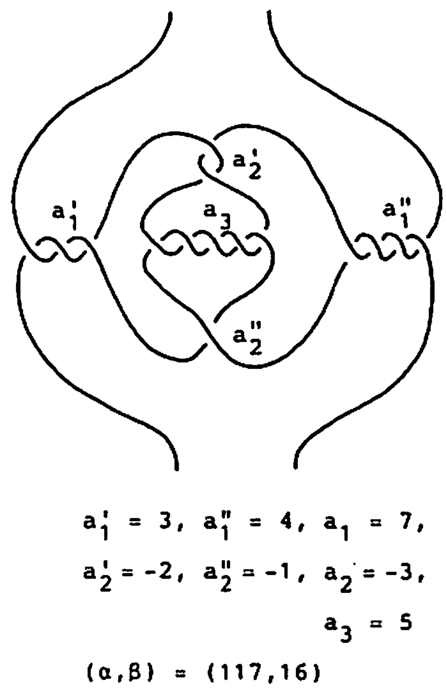


Fig. 0.2

Here r, b, α_i, and β_i are integers, such that $r \geq 0$, $\alpha_i \geq 2$ and $g.c.d.(\alpha_i, \beta_i) = 1$ $(1 \leq i \leq r)$. The integers b, a'_i, a''_i in Figures 0.1 and 0.2 denote numbers of right-hand halftwists. A box $\boxed{\alpha, \beta}$ stands for a rational tangle as illustrated in

Figure 0.2, where α and β are defined by the continued fraction

$$\beta/\alpha = (a_1 + (-a_2 + (a_3 + (\cdots(\pm a_n)^{-1}\cdots)^{-1})^{-1})^{-1})^{-1}$$

with $a_i = a_i' + a_i''$, together with the condition that α and β are relatively prime and $\alpha > 0$.

It was shown by Montesinos [M1] that the double branched cover M of S^3 branched along L is the Seifert fibred space $\{-b; (o_1, O); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ in the notation of [O]. The base orbifold of this Seifert fibred space is $S^2(\alpha_1, \cdots, \alpha_r)$, the 2-dimensional orbifold with underlying space S^2 and with cone points of local groups $Z_{\alpha_1}, \cdots, Z_{\alpha_r}$. The covering involution τ on M operates fibre-preserving; it reverses the orientation of the fibres and induces a reflection through an equator of the base orbifold, where all of the cone points lie on the equator. Since $(S^3, L) = (M, Fix(\tau))/\tau$, we may regard (S^3, L) as a 3-dimensional orbifold with underlying space S^3 and the singular set L of cone angle π; we denote it by the symbol $\mathcal{O}(L)$. Then by the preceding fact, $\mathcal{O}(L)$ has the structure of Seifert fibred orbifold over the 2-dimensional orbifold $D^2(\alpha_1, \cdots, \alpha_r)$, the 2-dimensional closed orbifold with underlying surface D^2 and corner reflectors of angle $\pi/\alpha_1, \cdots, \pi/\alpha_r$ in this order. The fibred orbifold structure $\bar{\eta} : \mathcal{O}(L) \to D^2(\alpha_1, \cdots, \alpha_r)$ is completely determined (up to orientation preserving equivalence) by the following data (see [BnS1, D1]).

(0.1) The ordered set $(\beta_1/\alpha_1, \beta_2/\alpha_2, \cdots, \beta_r/\alpha_r) \in (Q/Z)^r$ up to cyclic permutation and reversal of the order.

(0.2) The Euler number $e(\bar{\eta}) = \frac{1}{2}(b - \sum_{i=1}^r \beta_i/\alpha_i)$.

We call a Montesinos link L *spherical*, if the double cover M is a spherical manifold. Then the following properties are equivalent:

(i) L is spherical.

(ii) $\pi_1(M)$ is finite.

(iii) $2 - \sum_{i=1}^r (1 - 1/\alpha_i) > 0$, and L is not a trivial 2-component link.

(iv) Either $r \leq 2$ and L is a 2-bridge link which is not a trivial 2-component link, or $r = 3$ and $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, n)$ with $n \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$.

If a Montesinos link L is neither spherical nor a trivial 2-component link, then it is called *sufficiently complicated.* If $r \geq 3$, then the preceding data (0.1) and (0.2) form complete isotopy invariants of L (see [BnS2, BuZ, Z]); and if L is sufficiently complicated, then the symmetry group $Sym(S^3, L) = \pi_0 Diff(S^3, L)$ is isomorphic to $Out(\pi_1(\mathcal{O}(L)))$, the outer-automorphism group of the orbifold fundamental group $\pi_1(\mathcal{O}(L))$ of the orbifold $\mathcal{O}(L)$ (see [BiZ1, 2]). Using this fact, Boileau and Zimmermann [BiZ1] have calculated the symmetry groups of sufficiently complicated Montesinos links (cf. [BnS2]). For a spherical Montesinos link, the above result does not hold.

However a spherical Montesinos link L has a nice geometric feature; that is, the orbifold $\mathcal{O}(L)$ admits a unique spherical structure. In other words, there is an embedding of $\pi_1(\mathcal{O}(L))$ into the isometry group $IsomS^3$ of S^3, and $\mathcal{O}(L)$ is equivalent to the quotient orbifold of S^3 by the action of $\pi_1(\mathcal{O}(L))$; and such embeddings of $\pi_1(\mathcal{O}(L))$ into $IsomS^3$ is unique up to conjugation. Moreover, if we assume the orbifold uniformization theorem announced by Thurston [T2], then it follows that $Sym(S^3, L) \cong \pi_0 Isom\mathcal{O}(L)$ for a spherical Montesinos link L (Theorem 6.1).

The purpose of this paper is to describe explicitly the spherical structure of $\mathcal{O}(L)$ for a spherical Montesinos link L, and calculate $Isom\mathcal{O}(L)$ (Theorems 3.4 and 4.1). Moreover, we give a description of the geodesic link $\tilde{L}$ in S^3 which arises from L as $\tilde{L} = p^{-1}(L)$, where $p : S^3 \to S^3$ is the composition of the universal covering projection $S^3 \to M$ and the branched covering projection $M \to S^3$ (Theorem 5.3). For the case where L is a 2-bridge link, Burde [Bu2] has given a topological description of $\tilde{L}$, and he used their linking numbers to get a new proof of the classification of 2-bridge links [Bu1]. So we think these geodesic links are interesting objects.

This paper is organized as follows. In the first two sections, we review the theory of fibred orbifolds and the geometry of S^3 following the article of Scott [St], which serves a very comprehensive approach to the geometries of 3-manifolds. In [St], a nice argument is given which lists all possible finite subgroups of $IsomS^3$ acting freely on S^3 [St, Theorem 4.11]. But it does not give a sufficient condition. So we follow and complete the arguments given there, and recover the classification of such subgroups originally due to Hopf [H] and Threlfall-Seifert [TS]. In Sections 3, 4, and 5, we give precise descriptions of the spherical structure on $\mathcal{O}(L)$, the isometry group $Isom\mathcal{O}(L)$, and the geodesic link $\tilde{L}$. In the final section, we study the symmetry groups of spherical Montesinos links.

1. Fundamental facts about Seifert fibred orbifolds

In this section, we summarize fundamental facts concerning Seifert fibred orbifolds and fix notations. A Seifert fibred orbifold is a closed oriented 3-dimensional orbifold $\mathcal{O}^3$ together with a map η from $\mathcal{O}^3$ to a 2-dimensional orbifold $\mathcal{O}^2$ which has the following local property (cf. [BnS1, D1, T1]): Let U be a neighbourhood of a point x of $\mathcal{O}^2$, such that $(U, x) \cong (D^2, 0)/G_x$, where D^2 is the unit disk in $\mathbf{R}^2$ and G_x is a finite group acting on D^2 orthogonally. Then the action of G_x on D^2 lifts to a product action of G_x on $D^2 \times S^1$, so that the following diagram is commutative:

$$\begin{array}{ccc} \eta^{-1}(U) & = & (D^2 \times S^1)/G_x \\ \eta \downarrow & & p \downarrow \\ U & = & D^2/G_x \end{array}$$

これは orbit invariant であり, Seifert invariant でない!

Here we assume D^2, S^1, and $D^2 \times S^1$ are given canonical orientations and the upper isomorphism is orientation preserving.

If x is a singular point of $\mathcal{O}^2$, then the local invariant of the Seifert fibration η at x is defined as follows:

CASE 1. x is a cone point with $G_x \cong Z_\alpha$ $(\alpha \geq 2)$. Let g be the generator of G_x which acts on D^2 as rotation by $2\pi/\alpha$. Then g acts on S^1 as $2\pi\beta/\alpha$ -rotation for some integer β. Then the local invariant of η at x is defined to be (α, β), where β is well-defined modulo α.

CASE 2. x is a corner reflector with $G_x \cong D_\alpha$. Consider the subgroup $D_\alpha \cap SO(2) \cong Z_\alpha$, and let β be as above. Then the local invariant at x is (α, β).

The actual calculation of local invariants can be done by the following lemma.

LEMMA 1.1. *Let G be a finite group which acts effectively on $D^2 \times S^1$ as a product of rotations of D^2 and S^1. Let α be the order of the action of G on D^2-factor, and let g be an element of G which acts on D^2 as rotation by $2\pi/\alpha$. Then g acts on S^1-factor as rotation by $2\pi k/|G|$ for some integer k. Then the local invariant at $[0] \in D^2/G$ of the Seifert fibration $D^2 \times S^1/G \to D^2/G$ is equal to (α, β), where $\beta \equiv k \bmod \alpha$.*

PROOF. If $\alpha = |G|$, this lemma is nothing other than the definition of the local invariant. If $\alpha \neq |G|$, consider the normal subgroup G_0 of G consisting of the elements which act trivially on D^2-factor. Then $G/G_0 \cong Z_\alpha$ acts effectively on $(D^2 \times S^1)/G_0 \cong D^2 \times (S^1/G_0)$. Now the desired result follows from this fact.

There is another invariant for $\eta : \mathcal{O}^3 \to \mathcal{O}^2$, which is called the *Euler number* of η and denoted by $e(\eta)$. This invariant is determined by the following naturality property (see [BnS1, D1, NR, RV, St]).

LEMMA 1.2. *Let $\tilde{\eta} : \widetilde{\mathcal{O}^3} \to \widetilde{\mathcal{O}^2}$ and $\eta : \mathcal{O}^3 \to \mathcal{O}^2$ be Seifert fibred orbifold structures on oriented closed 3-dimensional orbifolds $\widetilde{\mathcal{O}^3}$ and $\mathcal{O}^3$ respectively. Let $\tilde{p} : \widetilde{\mathcal{O}^3} \to \mathcal{O}^3$ and $p : \widetilde{\mathcal{O}^2} \to \mathcal{O}^2$ be orbifold coverings such that $\eta\tilde{p} = p\tilde{\eta}$. Let $\tilde{d}$ [resp. d] be the homological degree of $\tilde{p}$ [resp. the geometric degree of p], and put $m = \tilde{d}/d$. [Thus a regular fiber of $\tilde{\eta}$ covers $|m|$ times a regular fiber of η.] Then we have $e(\eta) = (m/d)e(\tilde{\eta})$.*

As a special case of the classification theorem proved by [BnS1, D1], we have the following.

PROPOSITION 1.3. *Suppose η and η' be Seifert fibrations on oriented 3-*

dimensional orbifolds, such that their base orbifolds are diffeomorphic and their underlying surfaces have no more than one boundary component. Then if the local invariants of corresponding points are equal and the Euler numbers are equal, then η and η' are equivalent.

In this paper, we use the following notations and facts:

(1.4) $S(e;(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r))$ denotes the Seifert fibred orbifold with base orbifold $S^2(\alpha_1,\cdots,\alpha_r)$, Euler number e, and local invariants $(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r)$. Put $d_i = g.c.d.(\alpha_i,\beta_i)$, $\alpha_i' = \alpha_i/d_i$, and $\beta_i' = \beta_i/d_i$. Then the singular set consists of the singular fibers for which $d_i \geq 2$. The underlying space is the Seifert fibred manifold $S(e;(\alpha_1',\beta_1'),\cdots,(\alpha_r',\beta_r'))$. [If we use the notation of [O], it is described as $\{b';(o_1,O);(\alpha_1',\beta_1'),\cdots,(\alpha_r',\beta_r')\}$, where b' is an integer such that $e = -(b' + \sum_{i=1}^{r}\beta_i'/\alpha_i')$.]

(1.5) $D(e;(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r))$ denotes the Seifert fibred orbifold with base orbifold $D^2(\alpha_1,\cdots,\alpha_r)$, Euler number e, and local invariants $(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r)$. The underlying space is the 3-sphere S^3, and if $g.c.d.(\alpha_i,\beta_i) = 1$ for $1 \leq i \leq r$, then the singular set forms the Montesinos link $L(b;(\alpha_1,\beta_1),\cdots,(\alpha_r,\beta_r))$, where b is an integer determined by $e = \frac{1}{2}(b - \sum_{i=1}^{r}\beta_i/\alpha_i)$.

(1.6) $D^2(p;\alpha_1,\cdots,\alpha_r)$ denotes the 2-dimensional orbifold obtained from $D^2(\alpha_1,\cdots,\alpha_r)$ by adding a cone point of local group Z_p in the interior of the underlying disk. If $\mathcal{O}^3$ is a Seifert fibred orbifold over $D^2(p;\alpha_1,\cdots,\alpha_r)$ whose local invariant at the cone point is (p,q). Then the underlying space $|\mathcal{O}^3|$ is the lens space with $\pi_1(|\mathcal{O}^3|) \cong Z_{p'}$ where $p' = p/g.c.d.(p,q)$.

(1.7) Let $\eta : \mathcal{O}^3 \to \mathcal{O}^2$ be a Seifert fibration. Then η induces an epimorphism from $\pi_1(|\mathcal{O}^3|)$ to $\pi_1(|\mathcal{O}^2|)$.

2. The geometries of S^2 and S^3

Following [St] (cf. [M2]), we describe these geometries by using quaternions. Let H be the quaternion skew field, and identify S^n $(n = 1,2,3)$ with subspaces of H as follows.

$$S^3 = \{q \in H \mid |q| = 1\}, \tag{2.1}$$

$$S^2 = \{q \in H \mid |q| = 1, Re(q) = 0\}, \tag{2.2}$$

$$S^1 = \{z \in C \subset H \mid |z| = 1\}. \tag{2.3}$$

Here, for a quaternion $q = a + bi + cj + dk$ $(a,b,c,d \in R)$, $|q|$ denotes the norm $\sqrt{a^2+b^2+c^2+d^2}$, and $Re(q)$ denotes the real part a of q. Every quaternion q can be expressed as $q = z_1 + z_2 j$ $(z_1, z_2 \in C)$, and we regard C as a subspace of H by identifying $z \in C$ with $z+0j \in H$. The norm $|\ \ |$ induces a metric on H, and S^n $(n = 1,2,3)$ have induced metrics. The group S^3 acts on itself by conjugation and leaves S^2 invariant. Thus we obtain an epimorphism $\psi : S^3 \to Isom^+S^2$, with $Ker(\psi) =< -1 >$, by letting

$$\psi(q)(x) = qxq^{-1} \quad (q \in S^3, \quad x \in S^2). \tag{2.4}$$

Let a be the antipodal map on S^2, i.e. $a(q) = -q$. Then $IsomS^2$ is the direct product of $Isom^+S^2$ and the group $< a > \cong Z_2$. To understand the isometry $\psi(q)$, note that any unit quaternion q is written as

$$q = \cos\theta + q_0 \sin\theta, \tag{2.5}$$

where $\theta \in R$ and $q_0 \in S^2$. [Thus $Re(q_0) = 0$ and $Re(q) = \cos\theta$.] Then we see that $\psi(q)$ is the rotation of S^2 of angle 2θ with axis q_0. In particular if $q \in S^2$, then $\psi(q)$ is the π -rotation of S^2 with $Fix(\psi(q)) = \{\pm q\}$.

Now we give precise descriptions of regular polyhedral groups. Let P be a regular polyhedron inscribed in S^2, and let V_P, E_P, and F_P be the subsets of S^2 which are the images of the sets {the vertices}, {the centers of the edges}, and {the centers of the faces} under the projection $P \to S^2$ from the origin. We denote the subgroup of $Isom^+S^2 \cong SO(3)$ consisting of the elements which preserves a regular polyhedron P by the same symbol P. Then the binary polyhedral group $P^* = \psi^{-1}(P) \subset S^3$ consists of the elements of the form $\pm(\cos\frac{k\pi}{\alpha} + q\sin\frac{k\pi}{\alpha})$, where $q \in V_P \cup E_P \cup F_P$; k and α are integers such that $0 \le k < \alpha$, and α is the order of a vertex, 2, or the number of edges of a face according as q belongs to V_P, E_P, or F_P. We denote the tetrahedron, the octahedron, and the icosahedron by the symbols T, O, and I respectively.

PROPOSITION 2.6. *The following is the list of finite subgroups H of $Isom^+$ S^2 up to conjugation, the quotient orbifold S^2/H, and the normalizer $N(H^*)$ of $H^* = \psi^{-1}(H)$ in S^3.*

H	S^2/H	$N(H^*)$
Z_n $(n \ge 2)$	$S^2(n,n)$	D_S
D_n $(n \ge 2)$	$S^2(2,2,n)$	D^*_{2n} *if* $n > 2$ O^* *if* $n = 2$
T	$S^2(2,3,3)$	O^*
O	$S^2(2,3,4)$	O^*
I	$S^2(2,3,5)$	I^*

Here D_S denotes the subgroup $< S^1, j >$, and we assume $Z^*_n \subset S^1$ and $D_n \subset D_S$; thus $Z^*_n =< \omega >$ and $D^*_n =< \omega, j >$ where $\omega = e^{\pi i/n}$. We also assume $D^*_2 = \{\pm 1, \pm i, \pm j, \pm k\} \subset T^* \subset O^*$. Thus the vertex set V_O of the octahedron O is $\{\pm i, \pm j, \pm k\}$, and the tetrahedron is situated so that $E_T = V_O$. [See Figure 2.1, where we draw a cube which is a dual of O.]

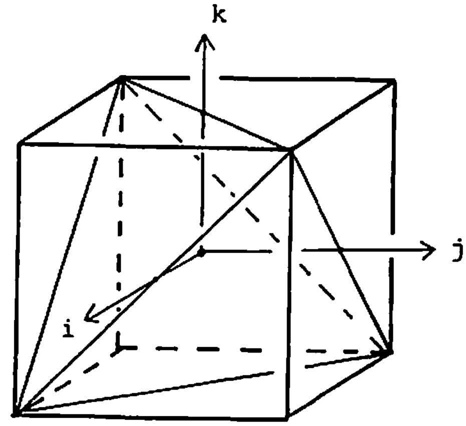


Fig. 2.1

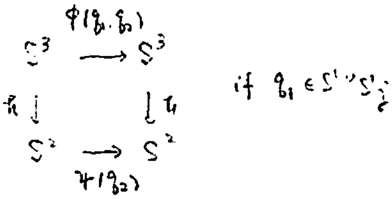
PROOF. The list of finite subgroups of $IsomS^2$ can be found in [Wo, Section 2.6]. If $H^* = Z_n^*$ $(n \geq 2)$, then the fixed point set of the action $H = \psi(H^*)$ on S^2 is $\{\pm i\}$. Thus if $q \in N(H^*)$, then $\psi(q)$ preserves $\{\pm i\}$. If $\psi(q)(i) = i$, then $q \in S^1$; and if $\psi(q)(i) = -i$, then $q \in S^1 j$. Hence $N(Z_n^*) \subset S^1 \cup S^1 j$. Thus we obtain $N(Z_n^*) = S^1 \cup S^1 j$, since the converse implication is trivial. If $H^* = D_n^*$ $(n \geq 3)$, then $\pm i$ are the only points of S^2 with isotopy group Z_n under the action of $\psi(D_n^*)$. Thus by the above argument, we see $N(D_n^*) \subset S^1 \cup S^1 j$. By considering the image of $j \in D_n^*$ under conjugations, we obtain $N(D_n^*) = D_{2n}^*$. If $H^* = D_2^*$, then it is clear that $O^* \subset N(D_2^*)$. Since D_2^* is characteristic in O^*, the proof of $N(D_2^*) = O^*$ is reduced to the proof of $N(O^*) = O^*$. Similarly the proof of $N(T^*) = O^*$ is reduced to the above. If $H^* = O^*$ [resp. I^*], then V_O [resp. V_I] is the set of points with isotopy group Z_4 [resp. Z_5] under the action of H. Hence any element of $N(H^*)$ preserves V_O [resp. V_I] and hence preserves the regular polyhedron O [resp. I]. Thus we have $N(H^*) = H^*$.

Next we describe the geometry of S^3. Let $\phi : S^3 \times S^3 \to Isom^+ S^3$ be the homomorphism defined by

$$\phi(q_1, q_2)(q) = q_1 q q_2^{-1}. \tag{2.7}$$

Then ϕ is an epimorphism and $Ker\phi = \langle (-1,-1) \rangle \cong Z_2$. Let c be the orientation reversing isometry of S^3 defined by $c(q) = q^{-1}$. Then $IsomS^3$ is a semi-direct product of $Isom^+ S^3$ and the group $\langle c \rangle \cong Z_2$; moreover we have

$$c\phi(q_1, q_2)c^{-1} = \phi(q_2, q_1). \tag{2.8}$$



Let $h : S^3 \to S^2$ be a map defined by $h(q) = q^{-1}iq$. Then h gives the Hopf fibration; the fiber $h^{-1}(h(q))$ through q is equal to $S^1q = \phi(S^1, 1)q$.

LEMMA 2.9. (1) *An isometry $\phi(q_1, q_2)$ preserves h, iff $q_1 \in D_S = S^1 \cup S^1j$. If $q_1 \in S^1$ [resp. $q_1 \in S^1j$], $\phi(q_1, q_2)$ preserves [resp. reverses] the fibre orientation, and the action on the base space S^2 induced by $\phi(q_1, q_2)$ is $\psi(q_2)$ [resp. $\psi(q_2)$a].*

(2) *An isometry $\phi(q_1, q_2)$ $(q_1 \in D_S)$ preserves the typical fiber $h^{-1}(i) = S^1$, iff $(q_1, q_2) \in (S^1 \times S^1) \cup (S^1j \times S^1j)$. Moreover there is a (closed) tubular neighbourhood V of $S^1 = h^{-1}(i)$, such that there is an orientation preserving diffeomorphism $V \cong D^2 \times S^1$ with D^2 the unit disk in C, which satisfies the following condition: Let (ω_1, ω_2) be an element of $S^1 \times S^1$. Then the action on $V \cong D^2 \times S^1$ induced by $\phi(\omega_1, \omega_2)$ is $L(\omega_2^2) \times L(\omega_1\bar{\omega}_2)$. Here $L(\omega)$ $(\omega \in S^1)$ denotes a restriction of the map on C induced by multiplication by ω.*

PROOF. (1) can be proved by direct calculation.

(2) Note that the Hopf fibration h is equivalent to the fibration $\hat{h} : S^3 \to C \cup \{\infty\}$ defined by $\hat{h}(z_1 + z_2j) = z_2/z_1$. Let ϵ be a small positive real number and put $D_\epsilon = \{z \in C \mid |z| \leq \epsilon\}$ and $V = \hat{h}^{-1}(D_\epsilon)$. Then V is a tubular neighbourhood of $S^1 = \hat{h}^{-1}(0)$, and we can find an orientation preserving diffeomorphism $f : V \to D_\epsilon \times S^1$ which satisfies the following conditions:

(i) $p \circ f = \hat{h}|_V$, where p is the projection $D_\epsilon \times S^1 \to D$.

(ii) $f(z + 0j) = (0, z)$ for any $z \in S^1$.

Now the desired formula follows from the following identities.

$$\phi(\omega_1, \omega_2)(z_1 + z_2j) = \omega_1\overline{\omega_2}z_1 + \omega_1\omega_2z_2j,$$

$$\hat{h}(\phi(\omega_1, \omega_2)(z_1 + z_2j)) = \omega_2^2\hat{h}(z_1 + z_2j).$$

We now describe the finite subgroups of $IsomS^3$ acting freely on S^3 following the arguments of [St]. Let G be such a subgroup, and put $\tilde{G} = \phi^{-1}(G)$, $H_i^* = p_i(\tilde{G}) < S^3$ $(i = 1, 2)$, where p_i is the projection of $S^3 \times S^3$ to the i-th factor. Since $\tilde{G}$ is conjugate to a subgroup of $S^1 \times S^3$ [St, Theorem 4.10], we may assume $H_1^* = Z_n^* < S^1$ for some integer n. Hence, by Lemma 2.9, the Hopf fibration h induces a Seifert fibration η on $M = S^3/G$ over the 2-orbifold $\mathcal{O}^2 = S^2/\psi(H_2^*)$.

THEOREM 2.10. [H, TS]. *The following is a list of finite subgroups G of $IsomS^3$ acting freely on S^3 up to conjugation, and the quotient Seifert fibred spaces $M = S^3/G$.*

Type 1. *$G = \phi < (\omega^{q+1}, \omega^{q-1}) >$ with $\omega = e^{\pi i/p}$. Here p is a positive integer and q is an integer relatively prime to p. In this case M is the Lens space*

$L(p,q)$, and the Seifert fibration η is given by $S(d_2/p_2;(p_2,-r_2),(p_2,qr_2))$ where $d_2 = g.c.d.(p,q-1)$, $p_2 = p/d_2$, $q_2 = (q-1)/d_2$, and $q_2r_2 \equiv 1 \bmod p_2$.

Type 2. *Either (2-i) $G = \phi(Z_m^* \times D_\alpha^*)$ where g.c.d.$(m, 2\alpha) = 1$, or (2-ii) G is a diagonal subgroup of index 2 in $\phi(Z_{2m}^* \times D_\alpha^*)$ where m is even and g.c.d.$(m,\alpha) = 1$: To be precise, let $\gamma_1 : Z_{2m}^* \to Z_2$ and $\gamma_2 : D_\alpha^* \to D_\alpha^*/Z_\alpha^* \cong Z_2$ be the natural epimorphisms, and put $\Delta(Z_{2m}^* \times D_\alpha^*) = Ker[\gamma_1 \times \gamma_2 : Z_{2m}^* \times D_\alpha^* \to Z_2]$. Then $G = \phi(\Delta(Z_{2m}^* \times D_\alpha^*))$. In these cases $M = S(-m/\alpha;(2,1),(2,1),(\alpha,\beta))$ where $\beta \equiv m \bmod \alpha$.*

Type 3. *Either (2-i) $G = \phi(Z_m^* \times T^*)$ where g.c.d.$(m,6) = 1$, or (3-ii) G is a diagonal subgroup of index 3 in $\phi(Z_{3m}^* \times T^*)$ where m is odd and $m \equiv 0 \bmod 3$: To be precise, let $\gamma_1 : Z_{3m}^* \to Z_3$ and $\gamma_2 : T^* \to Z_3$ be natural epimorphisms, and put $\Delta(Z_{3m}^* \times T^*) = ker[\gamma_1 \times \gamma_2 : Z_{3m}^* \times T^* \to Z_3]$. Then $G = \phi(\Delta(Z_{3m}^* \times T^*))$. In these cases, $M = S(-m/6;(2,1),(3,\beta_2),(3,\beta_3))$. Here $\beta_2 \equiv \beta_3 \equiv m \bmod 3$ if G is of type (3-i); and $\beta_2 \not\equiv \beta_3 \bmod 3$ if G is of type (3-ii).*

Type 4. *$G = \phi(Z_m^* \times O^*)$ where g.c.d.$(m,24) = 1$. In this case $M = S(-m/12;(2,1),(3,\beta_2),(4,\beta_3))$, where $\beta_2 \equiv m \bmod 3$ and $\beta_3 \equiv m \bmod 4$.*

Type 5. *$G = \phi(Z_m^* \times I^*)$ where g.c.d.$(m,60) = 1$. In this case $M = S(-m/30;(2,1),(3,\beta_2),(5,\beta_3))$, where $\beta_2 \equiv m \bmod 3$ and $\beta_3 \equiv m \bmod 5$.*

REMARK 2.11. (1) Suppose M is not of type 1, and let f be a regular fibre of M. Then the order of f in $\pi_1(M)$ is $2m$. If $M = S(-m/\alpha;(2,1),(2,1),(\alpha,\beta))$ with α odd, then the order of f in $H_1(M)$ is also $2m$. Otherwise it is m. In particular, the order of f in $H_1(M)$ is ≤ 2, iff $m = 1$, that is $H_1^* = Z_1^*$.

(2) There are slight errors in the statements of [O, p.112 Theorem 2 (ii) and (iii)]. They should be read as follows:

(ii) $M = \{b;(o_1,O);(2,1),(2,1),(\alpha_3,\beta_3)\}$. Let $m = (b+1)\alpha_3 + \beta_3$; if $(m,2\alpha_3) = 1$, then $\pi_1(M) = C_m \times D_{4\alpha_3}^*$ ($= Z_m \times D_{\alpha_3}^*$ in our notation), and if $m = 2^k m'$ $(k \geq 1, m' \equiv 1 \bmod 2)$, then we have $\pi_1(M) = C_{m'} \times D'_{2^{k+2}\alpha_3}$.

(iii) $M = \{b;(o_1,O);(2,1),(3,\beta_2),(3,\beta_3)\}$. Let $m = 6b+3+2(\beta_2+\beta_3)$; if $(m,12) = 1$ then $\pi_1(M) = C_m \times T^*$, and if $m = 3^k m'$ $(k \geq 1, m' \not\equiv 0 \bmod 3)$, then $\pi_1(M) = C_{m'} \times T'_{8\cdot 3^{k+1}}$.

Proof of Theorem 2.10 continuing [St, Section 4]. In [St, Theorem 4.11], a detailed list of all possible finite subgroups of $IsomS^3$ acting freely on S^3 is given. What we have to do is to find a explicit condition for a group G in the list to act freely on S^3, and to identify the quotient manifold. Every group G in the list is of the form as described just before Theorem 2.10, and the *orbifold* $\mathcal{M} = S^3/G$ admits a Seifert fibration η which is of the form (1.4). We explain how to obtain the local invariants and the Euler number of η through an example. Let G be a diagonal subgroup of index 3 in $\phi(Z_{3m}^* \times T^*)$ where m is odd; this belongs to Scott's list. Then the base orbifold $\mathcal{O}^2$ of the corresponding Seifert fibration is $S^2/\psi(T^*) = S^2(2,3,3)$. The two cone points of $\mathcal{O}^2$ with local group Z_3 are the images of V_T and $F_T(\subset S^2)$ respectively. Pick a point v of V_T; then $-v$ belongs to F_T. For $\epsilon = \pm 1$ let $T^*(\epsilon v) = \{q \in T^* \mid \psi(q)(\epsilon v) = \epsilon v\}$ and $G(\epsilon v) = \{g \in G \mid g(h^{-1}(\epsilon v)) = h^{-1}(\epsilon v)\}$. Then we see $T^*(v) = T^*(-v) \cong Z_3^*$ and $G(v) = G(-v) = \{\phi(q_1, q_2) \in G \mid q_2 \in T^*(v)\}$. By applying a suitable conjugation, we may assume $v = i$, $T^*(\pm v) = T^*(\pm i) =< \omega_1 >$, and $G(\pm v) = G(\pm i) =< \phi(\omega_1^3, 1), \phi(\omega_1, \omega_2) >$, where $\omega_1 = e^{\pi i/3m}$ and $\omega_2 = e^{\pi i/3}$. By Lemma 2.9, there is a closed tubular neighbourhood V of $h^{-1}(v)$ and an orientation preserving diffeomorphism $V \cong D^2 \times S^1$, such that the action of $\phi(\omega_1^3, 1)$ and $\phi(\omega_1, \omega_2)$ on V corresponds to $id \times L(\omega_1^3)$ and $L(\omega_2^2) \times L(\omega_1 \bar{\omega}_2)$ respectively. Note that $|G(v)| = 6m$ and $L(\omega_2^2) \times L(\omega_1 \bar{\omega}_2) = L(e^{2\pi i/3}) \times L(e^{2\pi i(1-m)/6m})$. Hence, by Lemma 1.1, the local invariant of η at $[v] \in \mathcal{O}^2$ is $(3, \beta_+)$ where $\beta_+ \equiv 1 - m \bmod 3$. To obtain the local invariant at $[-v] \in \mathcal{O}^3$,consider a conjugation by $\phi(1, j)$. Then it maps the fiber $h^{-1}(-v) = h^{-1}(-i)$ to $h^{-1}(i)$, and maps the group $G(-v)$ to $< \phi(\omega_1^3, 1), \phi(\omega_1, \bar{\omega}_2) >$. Thus, again by using Lemmas 1.1 and 2.9, we see that the local invariant of η at $[-v] \in \mathcal{O}^2$ is $(3, \beta_-)$ where $\beta_- \equiv -1 - m \bmod 3$. Similarly, we see that the local invariant at the cone point with local group Z_2 is $(2, \beta)$, where $\beta \equiv m \bmod 2$; so it is (2,1). Hence we see the Seifert fibred orbifold $\mathcal{M}$ is $S(e; (2,1), (3, \beta_+), (3, \beta_-))$ where $\beta_\pm = \pm 1 - m \bmod 3$, and e is obtained later. The singular point of this orbifold is empty, iff $\beta_\pm \not\equiv 0 \bmod 3$, that is, $m \equiv 0 \bmod 3$. This is a necessary and sufficient condition for G to act freely on S^3. Finally we calculate the Euler number e. To do this, consider the following commutative diagram of fibred orbifolds:

$$\begin{array}{ccc} S^3 & \xrightarrow{\tilde{p}} & S^3/G \\ h \downarrow & & \eta \downarrow \\ S^2 & \xrightarrow{p} & S^2/\psi(T^*) \end{array}$$

Here $\tilde{p}$ and p are the projections, and we see that the degrees of $\tilde{p}$ and p are $|G| = 24m$ and $|T| = 12$ respectively. Hence by Lemma 1.2, we have $e(\eta) = (2m/12)e(h) = -m/6$. This completes the analysis for a diagonal subgroup of index 3 in $\phi(Z_{3m}^* \times T^*)$. Other groups can be treated similarly.

3. The spherical structures on spherical Montesinos links

Let L be a spherical Montesinos link, and recall the orbifold $\mathcal{O}(L)$ (see Introduction). In this section, we show that $\mathcal{O}(L)$ admits a unique spherical structure. Since the double branched cover M of S^3 branched along L admits a unique spherical structure, it suffices to show that there is an order 2 element τ of $Isom^+ M$

unique up to conjugation, such that $(M, Fix(\tau))/\tau \cong (S^3, L)$. To do this we identify M with S^3/G, where G is a subgroup of $Isom S^3$ as described in Theorem 2.10 and we use the notations given there. Thus M is the Seifert fibred space $S(e;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$, where $(\alpha_1,\alpha_2,\alpha_3)$ is equal to $(1,p_2,p_2)$, $(2,2,\alpha)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$, according as G is of type 1, 2, 3, 4, or 5.

First we show the existence of such an isometry τ. By (1.5), it suffices to find an isometry τ of M which satisfies the following two conditions:

(3.1) τ preserves the Seifert fibration $\eta : M \to \mathcal{O}^2$, and reverses the fibre-orientations.

(3.2) Let $\bar{\tau}$ be an involution on $\mathcal{O}^2$ induced by τ. Then $\mathcal{O}^2/\bar{\tau}=D^2(\alpha_1,\alpha_2,\alpha_3)$.

Let $\tilde{\tau}$ be a lift of τ to the universal cover S^3 of M such that $Fix(\tilde{\tau}) \neq \emptyset$, and let (q_1,q_2) be an element of $S^3 \times S^3$ such that $\tilde{\tau} = \phi(q_1,q_2)$. Then we have

(3.3) $q_1, q_2 \in S^2 \subset S^3$, and $(q_1,q_2) \in \mathcal{N}(\tilde{G})$. Here $\mathcal{N}(\tilde{G})$ denotes the normalizer of $\tilde{G}$ in $S^3 \times S^3$.

This can be seen as follows: Since $\tilde{\tau}^2 = 1$, we have $(q_1^2, q_2^2) = \pm(1,1)$; on the other hand, since $Fix(\tilde{\tau}) \neq \emptyset$, we see q_1 and q_2 are conjugate in S^3. These implies $q_1^2 = q_2^2 = -1$, since $\tilde{\tau} \neq id$. This is equivalent to the condition $q_1, q_2 \in S^2$ (cf. (2.2)). The condition $(q_1,q_2) \in \mathcal{N}(\tilde{G})$ is equivalent to the condition that $\phi(q_1,q_2)$ is a lift of an isometry of M. By Lemma 2.9 (1), the conditions (3.1) and (3.2) are equivalent to the following conditions:

(3.1)′ $q_1 \in S^1 j$.

(3.2)′ $S^2/ < \psi(H_2^*), \psi(q_2)a > \cong D^2(\alpha_1,\alpha_2,\alpha_3)$.

Note that $e^{i\theta}j = e^{i\theta/2}je^{-i\theta/2}$ and $H_1^* = Z_n^*$. Hence there is a conjugation of S^3 acting trivially on Z_n^*, which sends q_1 to j. Thus (3.1)′ may be replaced by the condition that $q_1 = j$. The condition (3.2)′ is equivalent to the following condition: The involution $\psi(q_2)a$ preserves each of the orbits of the singular points under the action of $\psi(H_2^*)$ on S^2. [For example, if G is of type $3 \sim 5$ and H_2 is a polyhedral group P, then it requires that $\psi(q_2)a$ preserves the subsets V_P, E_P, and F_P of S^2.] Thus we see that condition (3.2)′ is equivalent to the following condition according as the type of G:

Case 1. G is of type 1.

(a) If $p_2 = 1$, then $q_2 \in S^2$.

(b) If $p_2 \geq 2$, then $q_2 \in S^1 j$.

Case 2. G is of type 2 and $H_2^* =< \omega_2, j >$, where $\omega_2 = e^{\pi i/\alpha}$ $(\alpha \geq 2)$.

(a) If $\alpha = 2$, then $q_2 \in V_O$.

(b) If α is even and $\alpha \geq 4$, then $q_2 \in Z_\alpha^* j$.

(c) If α is odd, then $q_2 \in Z_\alpha^* \sqrt{\omega_2} j$, where $\sqrt{\omega_2} = e^{\pi i/2\alpha}$.

Case 3. G is of type 3, then $q_2 \in E_O$

CASE 4. G is of type 4, then $q_2 \in V_O \cup E_O$

CASE 5. G is of type 5, then $q_2 \in E_I$.

Further, it is easily checked that if q_2 satisfies the above condition, then $\phi(q_1, q_2) = \phi(j, q_2)$ belongs to $\mathcal{N}(\tilde{G})$. Thus $\tilde{\tau} = \phi(j, q_2)$ satisfies all demanded conditions, and it gives the desired isometry τ of M. Moreover the conjugacy class of τ in $IsomM$ determined by $\phi(j, q_2)$ does not depend on a choice of q_2.

Next, we prove the uniqueness of τ. In fact we prove that if ν is an order 2 element of $Isom^+M$ such that M/ν is a homology sphere, then ν is conjugate to an isometry which satisfies the conditions (3.1) and (3.2). Then it follows that ν is conjugate to the isometry τ just constructed. Let ν be such an isometry. Since M/ν is a homology sphere, we see $Fix(\nu) \neq \emptyset$, and hence there is a lift $\tilde{\nu}$ of ν to the universal cover S^3 of M, such that $Fix(\tilde{\nu}) \neq \emptyset$. Let (q_1, q_2) be an element of $S^3 \times S^3$, such that $\tilde{\nu} = \phi(q_1, q_2)$; then it satisfies the condition (3.3). We show the condition (3.1)$'$ is achieved. Recall that $H_1^* = Z_n^*$. Suppose $n = 1$, then since $q_1 \in S^2$, there is a conjugation of S^3 acting trivially on H_1^* which sends q_1 to j. Thus we may assume q_1 satisfies (3.1)$'$. Suppose $n \geq 2$, then $q_1 \in \mathcal{N}(Z_n^*) = S^1 \cup S^1 j$ (cf. Proposition 2.6). To treat this case we use the fact that $\nu_* = -id$ where ν_* is the isomorphism of $H_1(M)$ induced by ν (cf. [Br, p.119 (2.2)]). Suppose (3.1)$'$ is not satisfied; i.e. $q_1 \in S^1$. Then τ preserves the fibre-orientations of η by Lemma 2.9 (1), and hence the order of the regular fiber in $H_1(M)$ is equal to 1 or 2. Thus if M is not a lens space, we have $n = 1$ by Remark 2.11 (1), a contradiction. Suppose M is a lens space $L(p, q)$, and recall the Seifert fibration η given in Theorem 2.10 (1). If $p_2 = 1$, then $H_1(M)$ is generated by a regular fibre, and therefore $H_1(M) = 0$ or Z_2. Then we see $(p, q) = (1, 0)$ or $(2, 1)$, and $H_1^* =< \omega^{q+1}, -1 >= Z_1^*$. [Recall the notations in Theorem 2.10.] Thus we have $n = 1$ and a contradiction. If $p_2 \geq 2$, then consider the homology classes s_1 and s_2 represented by the singular fibres. Here the orientations of the singular fibres are inherited from those of fibres of the Hopf fibration. Then both s_1 and s_2 are generators of $H_1(M) \cong Z_p$ and we have $s_2 = qs_1$. Since $\nu_* = -id$, we see $p = 1$ or 2 if $\nu_*(s_1) = s_1$, and $q \equiv -1 \bmod p$ if $\nu_*(s_1) = s_2$. In either case, $H_1^* =< \omega^{q+1}, -1 >= Z_1^*$; a contradiction. Thus we have proved that q_1 satisfies (3.1)$'$. Finally we show (3.2)$'$ is achieved. Suppose (3.2)$'$ is not satisfied. Then, for the 2-orbifold $\mathcal{O}^2/\tilde{\nu}$, we see either $|\mathcal{O}^2/\tilde{\nu}| \cong RP^2$ or $\mathcal{O}^2/\tilde{\nu}$ is isomorphic to $D(p_2; -)$, $D(\alpha; 2)$, or $D(2; 3)$. In any case M/ν has a nontrivial homology by (1.6) and (1.7). Thus we have proved the desired result. By noting (1.4), we obtain the following.

THEOREM 3.4. *Let L be a spherical Montesinos link. Then the orbifold $\mathcal{O}(L)$ admits a unique spherical structure. Moreover, the spherical structure is given by* *the following subgroup* $\tilde{\Gamma} < S^3 \times S^3$, *for which we have* $\phi(\tilde{\Gamma}) \cong \pi_1(\mathcal{O}(L))$ *and* $S^3/\phi(\tilde{\Gamma}) \cong \mathcal{O}(L)$.

Type 1. *L is a 2-bridge link of type (p,q). Then* $\tilde{\Gamma} = \langle (\omega^{q+1}, \omega^{q-1}), (j,j) \rangle$ *where* $\omega = e^{\pi i/p}$.

Type 2. $L = L(-b;(2,1),(2,1),(\alpha,\beta))$. *Put* $m = (b+1)\alpha + \beta$ *and* $\sqrt{\omega_2} = e^{\pi i/2\alpha}$.

(i) *Suppose* g.c.d.$(m, 2\alpha) = 1$. *Then*

$$\tilde{\Gamma} = \begin{cases} \langle Z_m^* \times D_\alpha^*, (j, \sqrt{\omega_2} j) \rangle & \text{if } \alpha \text{ is odd,} \\ \langle Z_m^* \times D_\alpha^*, (j,j) \rangle & \text{if } \alpha \text{ is even.} \end{cases}$$

(ii) *Suppose m is even and* g.c.d.$(m,\alpha) = 1$. *Then* $\tilde{\Gamma} = \langle \Delta(Z_{2m}^* \times D_\alpha^*), (j, \sqrt{\omega_2} j) \rangle$.

Type 3. $L = L(-b;(2,1),(3,\beta_2),(3,\beta_3))$. *Put* $m = 6b + 3 + 2(\beta_2 + \beta_3)$, *and let q_2 be an element of* E_O.

(i) *Suppose* g.c.d.$(m, 12) = 1$. *Then* $\tilde{\Gamma} = \langle Z_m^* \times T^*, (j, q_2) \rangle$.

(ii) *Suppose* $m \equiv 0 \bmod 3$ *and m is odd. Then* $\tilde{\Gamma} = \langle \Delta(Z_{3m}^* \times T^*), (j, q_2) \rangle$.

Type 4. $L = L(-b;(2,1),(3,\beta_2),(4,\beta_3))$. *Put* $m = 12b + 6 + 4\beta_2 + 3\beta_3$, *and let q_2 be an element of* $V_O \cup E_O$. *Then* $\tilde{\Gamma} = \langle Z_m^* \times O^*, (j, q_2) \rangle$.

Type 5. $L = L(-b;(2,1),(3,\beta_2),(5,\beta_3))$. *Put* $m = 30b + 15 + 10\beta_2 + 6\beta_3$, *and let q_2 be an element of* E_I. *Then* $\tilde{\Gamma} = \langle Z_m^* \times I^*, (j, q_2) \rangle$.

REMARK 3.5. The above proof also shows that, $\mathcal{O}(L)$ admits two different Seifert fibrations if $H_1^* = Z_1^*$. In fact, $\tilde{\tau} = \phi(j, q_2)$ is conjugate to $\phi(i, q_2)$ in this case, and hence $\mathcal{O}(L)$ admits a Seifert fibration whose base orbifold is $S^2/ \langle \psi(H_2^*), \psi(q_2) \rangle$ (cf. (3.2)'). Precise description is given as follows (cf. (1.4) and [M2, pp.170-171]).

Type 2 case. $S(-1/2\alpha;(2,1),(2,0),(2\alpha, 1-\alpha))$ with α odd, or $S(-2/\alpha; (2,0),(2,0),(\alpha,2))$ with α even. Then L is the union of the $(2,\alpha)$ torus link and the "core of index α".

Type 3 case. $S(-1/12;(2,0),(3,1),(4,-1))$. Then L is the $(3,4)$ torus knot.

Type 4 case. $S(-1/6;(2,2),(3,2),(4,2))$. Then L is the union of the $(2,3)$ torus knot and the "core of index 2".

Type 5 case. $S(-1/15;(2,2),(3,2),(5,2))$. Then L is the $(3,5)$ torus knot.

PROPOSITION 3.6. *Suppose the exterior of a spherical Montesinos link L is a Seifert fibred space, then either L is a $(2,n)$ torus link or one of the links listed in Remark 3.5.*

PROOF. It is well-known that the exterior of a 2-bridge link is a Seifert fibred space, iff it is a $(2,n)$ torus link. Thus we may assume L is not of type 1. By [BuM], either (1) the Seifert fibration η on $E(L)$ extends to that on S^3, or (2) L is an "earring", that is, an unknotted circle together with a finite number of meridian loops. If (2) holds, then the double branched cover M of (S^3, L) is a connected sum of projective spaces; thus L is a trivial knot or a 2-component Hopf link. Suppose (1) holds. Then η lifts to a Seifert fibration $\tilde{\eta}$ on M which is preserved by the covering involution τ. Note that the base orbifolds of η and $\tilde{\eta}$ are orientable, and τ preserves fiber-orientations of $\tilde{\eta}$. Then the preceding argument using Remark 2.11 (1) shows $H_1^* = Z_1^*$, and we obtain the desired result.

4. The isometry group of $\mathcal{O}(L)$

In this section, we prove the following theorem:

THEOREM 4.1. *Let L be a spherical Montesinos link. Then the isometry group of the spherical orbifold $\mathcal{O}(L)$ is as follows according as the type of L (cf. Theorem 3.4).*

Type 1. *If $q \equiv \pm 1 \bmod p$, then*

$$Isom\mathcal{O}(L) \cong \begin{cases} Isom^+\mathcal{O}(L) \cong \begin{cases} S^1 \ltimes Z_2 & p : odd \geqq 3, \\ S^1 \ltimes (Z_2 \oplus Z_2) & p : even \geqq 2, \end{cases} \\ Isom^+\mathcal{O}(L) \ltimes Z_2 \cong \begin{cases} S^1 \ltimes D_4 & p = 2, \\ (S^1 \times S^1) \ltimes (Z_2)^3 & p = 1. \end{cases} \end{cases}$$

Suppose $q \not\equiv \pm 1 \bmod p$, then

$$Isom\mathcal{O}(L) \cong \begin{cases} Isom^+\mathcal{O}(L) \ltimes Z_2 \cong D_4 & \text{if } q^2 \equiv -1 \bmod p, \\ Isom^+\mathcal{O}(L) & \text{if } q^2 \not\equiv -1 \bmod p. \end{cases}$$

$$Isom^+\mathcal{O}(L) \cong \begin{cases} Z_2 \oplus Z_2 & \text{if } q^2 \not\equiv 1 \bmod p, \\ D_4 & \text{if } p \text{ is odd and } q^2 \equiv 1 \bmod p \\ & \text{or } p \text{ is even and } q^2 \equiv 1 \bmod 2p, \\ (Z_2)^3 & \text{if } p \text{ is even and } q^2 \equiv p+1 \bmod 2p. \end{cases}$$

Type 2 (i). *$Isom\mathcal{O}(L) = Isom^+\mathcal{O}(L)$ is given by*

	$\alpha \geq 3$	$\alpha = 2$
$m \neq 1$	$Z_2 \oplus Z_2$	$Z_2 \oplus D_3$
$m = 1$	$S^1 \ltimes Z_2$ *if α is odd,* $S^1 \ltimes (Z_2 \oplus Z_2)$ *if α is even.*	$S^1 \ltimes (Z_2 \oplus D_3)$

Type 2 (ii). $IsomO(L) = Isom^+O(L) \cong Z_2 \oplus Z_2$.

Type 3 (i). $IsomO(L) = Isom^+O(L) \cong \begin{cases} Z_2 \oplus Z_2 & if\ m \neq 1, \\ S^1 \ltimes Z_2 & if\ m = 1. \end{cases}$

Type 3 (ii). $IsomO(L) = Isom^+O(L) \cong Z_2$.

Types 4 and 5. $IsonO(L) = Isom^+O(L) \cong \begin{cases} Z_2 & if\ m \neq 1, \\ S^1 \ltimes Z_2 & if\ m = 1. \end{cases}$

PROOF. Put $\Gamma = \phi(\tilde{\Gamma}) = \pi_1(O(L)) < Isom^+S^3$. Since an isometry of $O(L)$ lifts to an isometry of the universal covering orbifold S^3 of $O(L)$, we have isomorphisms;

$$IsomO(L) \cong \mathcal{N}(\Gamma)/\Gamma, \quad Isom^+O(L) \cong \mathcal{N}^+(\Gamma)/\Gamma,$$

where $\mathcal{N}(\Gamma)$ [resp. $\mathcal{N}^+(\Gamma)$] is the normalizer of Γ in $IsomS^3$ [resp. $Isom^+S^3$]. By (2.8), we see $IsomO(L) = Isom^+O(L)$ if L is not of Type 1. Let $\mathcal{N}(\tilde{\Gamma})$ be the normalizer of $\tilde{\Gamma}$ in $S^3 \times S^3$.Then we have $Isom^+O(L) \cong \mathcal{N}(\tilde{\Gamma})/\tilde{\Gamma}$.

If L is not of Type 1, then $IsomO(L)$ can be calculated from the following lemma.

LEMMA 4.2. *The nomalizer $\mathcal{N}(\tilde{\Gamma})$ is given as follows according as the type of L. Here $D'_S =< S^1_{(j)}, i >$ with $S^1_{(j)} = \{\cos\theta + j\sin\theta \mid \theta \in R\}$.*

Type 2.

	$\alpha \geq 3$	$\alpha = 2$
$m \neq 1$	$D^*_{2m} \times D^*_{2\alpha}$	$D^*_{2m} \times O^*$
$m = 1$	$D'_S \times D^*_{2\alpha}$	$D'_S \times O^*$

Type 3 (i). $\mathcal{N}(\tilde{\Gamma}) = \begin{cases} D^*_{2m} \times O^* & if\ m \neq 1, \\ D'_S \times O^* & if\ m = 1. \end{cases}$

Type 3 (ii). $\mathcal{N}(\tilde{\Gamma}) =< \tilde{\Gamma}, (\sqrt{\omega_1}, \sigma) >$, *where* $\sqrt{\omega_1} = e^{\pi i/6m}$ *and* σ *is an element of* T^* *such that* $\gamma_2(\sigma) = 1 \in Z_3$.

Type 4. $\mathcal{N}(\tilde{\Gamma}) = \begin{cases} D^*_{2m} \times O^* & if\ m \neq 1, \\ D'_S \times O^* & if\ m = 1. \end{cases}$

Type 5. $\mathcal{N}(\tilde{\Gamma}) = \begin{cases} D^*_{2m} \times I^* & if\ m \neq 1, \\ D'_S \times I^* & if\ m = 1. \end{cases}$

PROOF. If L is of Type 4 or 5, then $\tilde{\Gamma} = D^*_m \times P^*$ where $P = O$ or I according as L is of Type 4 or 5. Thus $\mathcal{N}(\tilde{\Gamma}) = N(D^*_m) \times N(P^*)$ and we obtain the desired result by Proposition 2.6 and the fact that if $m \neq 1$ then $m > 2$. If L is of Type 2 or 3, then $\mathcal{N}(\tilde{\Gamma})$ is a subgroup of $N(\tilde{\Gamma}_1) \times N(\tilde{\Gamma}_2)$ where $\tilde{\Gamma}_i = pr_i(\tilde{\Gamma}) < S^3 (i = 1, 2)$; and we can obtain the desired result through case by case checking.

Next we consider the case where L is the 2-bridge link of type (p,q). Then

$$\tilde{G} =< (\omega^{q+1}, \omega^{q-1}), (-1,-1) >,$$
$$\tilde{\Gamma} =< (\omega^{q+1}, \omega^{q-1}), (j,j) >,$$

where $\omega = e^{\pi i/p}$.

First, we treat the case where $q \not\equiv \pm 1 \bmod p$. We consider the following subgroups of $Isom S^3$.

$$\mathcal{I}_0^+ = \phi < S^1 \times S^1, (j,j) >,$$
$$\mathcal{I}^+ = \phi(D_S \times D_S),$$
$$\mathcal{I} =< \phi(D_S \times D_S), c > .$$

LEMMA 4.3. *$\mathcal{N}^+(\Gamma)$ and $\mathcal{N}(\Gamma)$ are subgroups of $\mathcal{I}^+$ and $\mathcal{I}$ respectively.*

PROOF. Put $p_1 = p/g.c.d.(p,q+1)$ and $p_2 = p/g.c.d.(p,q-1)$. Then $p_1, p_2 > 1$ for $q \not\equiv \pm 1 \bmod p$. Since $\tilde{G}$ is "characteristic" in $\tilde{\Gamma}$, we see $\mathcal{N}(\tilde{\Gamma}) < \mathcal{N}(\tilde{G}) < N(Z_{p_1}^*) \times N(Z_{p_2}^*) < D_S \times D_S$ by Proposition 2.6. Thus $\mathcal{N}^+(\Gamma) < \mathcal{I}^+$. Let γ be an orientation reversing isometry of S^3 belonging to $\mathcal{N}(\Gamma)$. Put $\gamma_0 = \gamma c$, then $\gamma_0 = \phi(q_1, q_2)$ for some $(q_1, q_2) \in S^3 \times S^3$. By (2.8) and the above argument, we see $q_i \in D_S$. Thus we have $\gamma_0 \in \mathcal{I}^+$ and so $\gamma \in \mathcal{I}$.

By the above lemma, it suffices to calculate the normalizer of Γ in $\mathcal{I}$. To make calculations smoother, we identify S^3 with $\{(z_1,z_2) \in C^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ where (z_1, z_2) corresponds to $z_1 + z_2 j$, and introduce the following notations:

(4.4) $L(\omega_1, \omega_2)$, where $\omega_1, \omega_2 \in S^1$, denotes the isometry of S^3 which sends (z_1, z_2) to $(\omega_1 z_1, \omega_2 z_2)$. Then we see $\phi(\omega_1, \omega_2) = L(\omega_1 \bar{\omega}_2, \omega_1 \omega_2)$.

(4.5) $J = \phi(j,j)$. Then we have $JL(\omega_1, \omega_2)J^{-1} = L(\bar{\omega}_1, \bar{\omega}_2)$, and $J^2 = 1$.

(4.6) $J_1 = \phi(1,j)$. Then we have $J_1 L(\omega_1,\omega_2) J_1^{-1} = L(\omega_2, \omega_1)$, $J_1 J J_1^{-1} = J$, and $J_1^2 = L(-1,-1)$.

(4.7) $cL(\omega_1, \omega_2)c^{-1} = L(\bar{\omega}_1, \omega_2)$, $\quad cJc^{-1} = J$.

Then we have

$$G =< L(\omega^2, \omega^{2q}) > \text{ where } \omega = e^{\pi i/p},$$
$$\Gamma =< L(\omega^2, \omega^{2q}), J >,$$
$$\mathcal{I}_0^+ =< L(S^1 \times S^1), J >,$$
$$\mathcal{I}^+ =< L(S^1 \times S^1), J, J_1 >,$$
$$\mathcal{I} =< L(S^1 \times S^1), J, J_1, c > .$$

Let $\mathcal{N}_0^+(\Gamma), \mathcal{N}^+(\Gamma)$ and $\mathcal{N}(\Gamma)$ be the nomalizers of Γ in $\mathcal{I}_0^+$, $\mathcal{I}^+$ and $\mathcal{I}$ respectively.

LEMMA 4.8. (1) $\mathcal{N}_0^+(\Gamma) = < L(\omega, \omega^q), L(1,-1), L(-1,1), J >$

(2) $\mathcal{N}^+(\Gamma) = \begin{cases} < \mathcal{N}_0^+(\Gamma), J_1 > & \text{if } q^2 \equiv 1 \bmod p, \\ \mathcal{N}_0^+(\Gamma) & \text{otherwise.} \end{cases}$

(3) $\mathcal{N}(\Gamma) = \begin{cases} < \mathcal{N}^+(\Gamma), J_1 c > & \text{if } q^2 \equiv -1 \bmod p, \\ \mathcal{N}^+(\Gamma) & \text{otherwise.} \end{cases}$

PROOF. (1) This follows from the following fact. Let (ω_1, ω_2) be an element of $S^1 \times S^1$, then

$$\begin{aligned} & L(\omega_1,\omega_2)J \in \mathcal{N}(\Gamma) \\ & \Leftrightarrow L(\omega_1,\omega_2) \in \mathcal{N}(\Gamma), \\ & \Leftrightarrow L(\omega_1^2,\omega_2^2)J = L(\omega_1,\omega_2)JL(\omega_1,\omega_2)^{-1} \in \Gamma, \\ & \Leftrightarrow (\omega_1^2,\omega_2^2) \in < (\omega^2,\omega^{2q}) >, \\ & \Leftrightarrow (\omega_1,\omega_2) \in < (\omega,\omega^q),(1,-1),(-1,1) > . \end{aligned}$$

(2) By (4.6), we see

$$\begin{aligned} & J_1 \in \mathcal{N}(\Gamma), \\ & \Leftrightarrow L(\omega^{2q},\omega^2) = J_1 L(\omega^2,\omega^{2q})J_1^{-1} \in \Gamma, \\ & \Leftrightarrow (\omega^{2q},\omega^2) \in < (\omega^2,\omega^{2q}) >, \\ & \Leftrightarrow q^2 \equiv 1 \bmod p. \end{aligned}$$

Since $\mathcal{N}^+(\Gamma) \supsetneq \mathcal{N}_0^+(\Gamma)$ iff $J_1 \in \mathcal{N}(\Gamma)$, we obtain (2).

(3) By (4.7), we see $cL(\omega^2,\omega^{2q})c^{-1} = L(\bar{\omega}^2,\omega^{2q}) \notin < L(\omega^2,\omega^{2q}) >$, since $q \not\equiv \pm 1 \bmod p$. So $c \notin \mathcal{N}(\Gamma)$. By (4.6) and (4.7), we see

$$\begin{aligned} & J_1 c \in \mathcal{N}(\Gamma), \\ & \Leftrightarrow J_1 cL(\omega^2,\omega^{2q})c^{-1}J_1^{-1} = L(\omega^{2q},\bar{\omega}^2) \in < L(\omega^2,\omega^{2q}) >, \\ & \Leftrightarrow q^2 \equiv -1 \bmod p. \end{aligned}$$

From the above facts, we obtain (3).

Now $IsomO(L)$ can be calculated from the above lemma. More precisely, its generators and the actions of the generators are given as follows. Here the last column represents the image of (S^3, L) by the action of corresponding generators. Note that L has one or two components according as p is odd or even; and in case p is even, K_1 and K_2 denote the components of L which are suitably oriented (cf. [BuZ, Sb]).

Subcase 1. p is odd.

	Isom$\mathcal{O}(L)$	Generator	Action
$q^2 \not\equiv \pm 1 \bmod p$	$Z_2 \oplus Z_2$	$L(1,-1)$ $L(-1,1)$	$(S^3,-L)$ $(S^3,-L)$
$q^2 \equiv 1 \bmod p$	D_4	J_1 $L(1,-1)$	(S^3,L) $(S^3,-L)$
$q^2 \equiv -1 \bmod p$	D_4	$L(1,-1)J_1 c$ $L(1,-1)$	$(-S^3,L)$ $(S^3,-L)$

Subcase 2. p is even.

	Isom$\mathcal{O}(L)$	Generator	Action
$q^2 \not\equiv \pm 1 \bmod p$	$Z_2 \oplus Z_2$	$L(\omega,\omega^q)$ $L(1,-1)$	(S^3,K_2,K_1) $(S^3,-K_1,-K_2)$
$q^2 \equiv 1 \bmod 2p$	$Z_2 \oplus Z_2 \oplus Z_2$	J_1 $L(\omega,\omega^q)$ $L(1,-1)$	(S^3,K_1,K_2) (S^3,K_2,K_1) $(S^3,-K_1,-K_2)$
$q^2 \equiv p+1 \bmod 2p$	D_4	$L(\omega,\omega^q)J_1$ $L(\omega,\omega^q)$	$(S^3,-K_2,K_1)$ (S^3,K_2,K_1)
$q^2 \equiv -1 \bmod p$	D_4	$L(\omega,\omega^q)J_1 c$ $L(\omega,\omega^q)$	$(-S^3,K_2,-K_1)$ (S^3,K_2,K_1)

Finally, we treat the case where $q \equiv \pm 1 \bmod p$. It suffices to consider the case $q \equiv 1 \bmod p$. Then $\tilde{\Gamma} =< (1,\omega),(j,j) >$ with $\omega = e^{\pi i/p}$. We can see the following;

$$\mathcal{N}(\tilde{\Gamma}) = \begin{cases} < S^1_{(j)} \times Z^*_{2p}, (j,j) > & \text{if } p : \text{odd} \geq 3, \\ < S^1_{(j)} \times S^1_{(j)}, (i,i) > & \text{if } p = 1, \\ < S^1_{(j)}, i > \times D^*_{2p} & \text{if } p : \text{even}, \end{cases}$$

Then $Isom\mathcal{O}(L)$ can be calculated from the above and the fact that $\mathcal{O}(L)$ admits an orientation reversing isometry, iff $p = 1, 2$.

5. The geodesic link $\tilde{L}$

In this section we give a description of the geodesic link $\tilde{L}$ in S^3 which is obtained from a spherical Montesinos link L through the procedure explained in the introduction. To do this, we use the description of the space of all geodesics in S^3 given by [GW]. Note that the quaternion skew field H has the structure of the 4-dimensional metric vector space spanned by the orthonormal vectors 1, i, j and k. Each oriented geodesic in S^3 determines an oriented two plane through the origin in H, that is, a point in the Glassmann manifold $\tilde{G}_2(R^4)$ of oriented 2-planes in $H \cong R^4$. Let P be an oriented 2-plane in R^4, and let $\{u, v\}$ be an ordered orthonormal base of P. Then the exterior product $u \wedge v$ does not depend on the choice of $\{u, v\}$, and we denote it by the symbol ω_P. Then the correspondence $P \longmapsto \omega_P$ determines an embedding $\tilde{G}_2(R^4) \hookrightarrow \Lambda^2 R^4$. Let E_+ [resp. E_-] be the subspace of $\Lambda^2 R^4$ which is the $+1$ [resp. -1] eigen space of the $*$ operator (cf. [GW, p.115]). For each element q of $S^2 \subset H \cong R^4$, let e_q^+ and e_q^- be the elements of $\Lambda^2 R^4$ defined by

$$e_q^+ = \frac{1}{2}\{(1 \wedge q) + *(1 \wedge q)\},$$

$$e_q^- = \frac{1}{2}\{(1 \wedge q) - *(1 \wedge q)\}.$$

Then E_+ and E_- are spanned by $\{e_i^+, e_j^+, e_k^+\}$ and $\{e_i^-, e_j^-, e_k^-\}$ respectively. Note that these six vectors are mutually orthogonal and have length $1/\sqrt{2}$. Let S_+^2 and S_-^2 be the sphere of radius $1/\sqrt{2}$ about the origin in E_+ and E_- respectively. Then the following holds ([GW Lemma 5.2]).

LEMMA 5.1. $\tilde{G}_2(R^4) \cong S_+^2 \times S_-^2$.

For example, the oriented plane P spanned by the ordered base $\{1, q\}$, where $q \in S^2 \subset H$ (c.f. (2.2)), corresponds to $(e_q^+, e_q^-) \in S_+^2 \times S_-^2$, since $1 \wedge q = e_q^+ + e_q^-$.

To describe the action of $Isom S^3$ on $\tilde{G}_2(R^4)$, we identify $S_\pm^2$ with the sphere $S^2 \subset H$ through the correspondence $S_\pm^2 \ni e_q^\pm \leftrightarrow q \in S^2$. Then we have

LEMMA 5.2. *Let (q_1, q_2) be an element of $S^3 \times S^3$. Then the following holds.*

(1) *Let $\phi_*(q_1, q_2)$ be the homeomorphism of $\tilde{G}_2(R^4) \cong S_+^2 \times S_-^2$ induced by $\phi(q_1, q_2)$. Then $\phi_*(q_1, q_2) = \psi(q_1) \times \psi(q_2)$.*

(2) *Express $q_i = \cos\theta_i + \sin\theta_i \hat{q}_i$ where $\hat{q}_i \in S^2$ for $i = 1, 2$. Then $\phi(q_1, q_2)$ has a nonempty fixed point set, iff $\theta_1 \equiv \theta_2 \bmod 2\pi$. Moreover, if this condition holds, then $Fix(\phi(q_1, q_2))$ is the geodesic represented by $\pm(\hat{q}_1, \hat{q}_2) \in S_+^2 \times S_-^2$.*

PROOF. (1) Let $\hat{\phi}(q_1, q_2)$ be the linear map on $\Lambda^2 R^4$ which is induced from the linear map $\phi(q_1, q_2)$ on $R^4 \cong H$ that is defined by (2.7). Since $\phi(q_1, q_2)$ is orthogonal and orientation-preserving, $\hat{\phi}(q_1, q_2)$ is orthogonal and preserves the eigen spaces E_+ and E_-. Thus $\phi_*(q_1, q_2)$ is a product of orthogonal maps of S_+^2 and S_-^2. We prove (i) $\phi_*(q, q) = \psi(q) \times \psi(q)$, and (ii) the action of $\phi_*(q, 1)$ [resp. $\phi_*(1, q)$] to S_-^2-factor [resp. S_+^2-factor] is the identity. Then (1) follows from these facts. For $u \in S^2$, we have $\hat{\phi}(q,q)(1 \wedge u) = q1q^{-1} \wedge quq^{-1} = 1 \wedge quq^{-1}$. From this identity we have (i). To prove (ii), let Ψ be the homomorphism $S^3 \to IsomS^2_-$ defined by $\Psi(q) =$ the action of $\phi_*(q,1)$ to S^2_-. Then we have only to show that $Ker\Psi \supsetneqq \{\pm 1\}$. We show $i \in Ker\Psi$. By direct calculation, we see $\hat{\phi}(i,1)$ fixes e_i^-, e_j^- and e_k^-. Then it follows that the restriction of $\hat{\phi}(i,1)$ to E_- is the identity, and therefore $i \in Ker\Psi$. This completes the proof of (1).

(2) There are elements u_1 and u_2 of S^3 such that $q_1 = u_1 e^{i\theta_1} u_1^{-1}$ and $q_2 = u_2 e^{i\theta_2} u_2^{-1}$. Then $\phi(q_1,q_2) = \phi(u_1,u_2)\phi(e^{i\theta_1},e^{i\theta_2})\phi(u_1,u_2)^{-1}$. We see (i) $\phi(e^{i\theta_1},e^{i\theta_2})$ has a fixed point, iff $\theta_1 \equiv \theta_2 \bmod 2\pi$, and (ii) if $\theta_1 \equiv \theta_2 \bmod 2\pi$, then $Fix(\phi(e^{i\theta_1}, e^{i\theta_2}))$ is the (unoriented) geodesic represented by $\pm(i,i)$. Now the assersion follows from (1).

Let L be a spherical Montesinos link and $\tilde{L}$ be the corresponding geodesic link in S^3. Then each component of $\tilde{L}$ is the fixed point of an isometry $\tilde{\tau}$ of oder 2 which is a lift of the covering involution τ of the double branched cover of (S^3,L). Suppose $\tilde{\tau} = \phi(q_1,q_2)$, then q_1 and q_2 are elements of $S^2 \subset S^3$ by (3.3), and the geodesic $Fix(\tilde{\tau})$ is represented by $\pm(q_1,q_2) \in S^2_+ \times S^2_-$ by Lemma 5.2. Hence the geodesic link $\tilde{L}$ is the union of geodesics which are represented by the subset $\tilde{\Gamma} \cap (S^2 \times S^2)$ of $S^2_+ \times S^2_- \cong \tilde{G}_2(R^4)$. This set can be obtained by the argument of Section 3. To state the result, we need the following notations. For a natural number m, let $C_m = \{\omega^r j \mid 0 \le r \le m-1\}$, where $\omega = e^{2\pi i/m}$. S^0 denotes $\{k, -k\}$. For the octahedron O, we consider the decomposition of E_O into the disjoint union of the subsets $E_O^{(r)}$ $(r = 0,1,2)$ as illustrated in Figure 5.1.

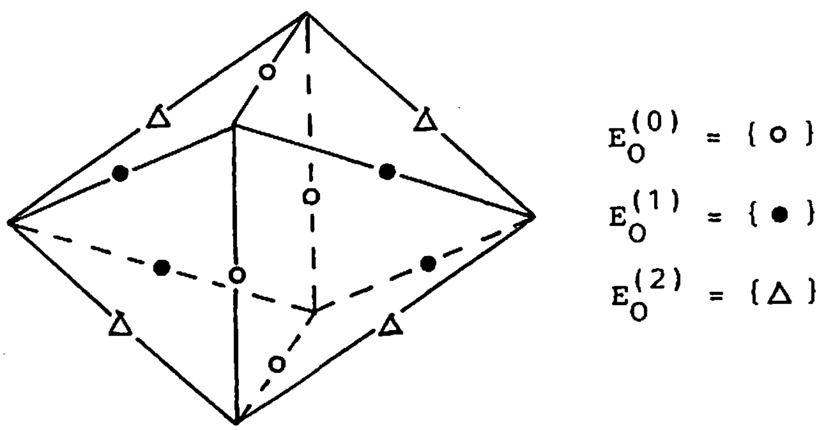


Fig. 5.1

THEOREM 5.3. *Let L be a spherical Montesinos link. If we orient L suitably* *and give $\tilde{L}$ the induced orientation, then the oriented geodesic link $\tilde{L}$ is expressed as follows as subsets of $S^2_+ \times S^2_-$.*

Type 1. $\{(\omega_1^r j, \omega_2^r j) \mid 0 \le r \le p-1\}$, *where* (ω_1, ω_2) *is* $(e^{2\pi(q+1)i/p}, e^{2\pi(q-1)i/p})$ *or* $(e^{\pi(q+1)i/p}, e^{\pi(q-1)i/p})$ *according as p is odd or even.*

Type 2 (i). $C_m \times \{C_{2\alpha} \cup S^0\}$.

Type 2 (ii). $\{C_{2m} \times C_{2\alpha}\} \cup \{\omega C_{2m} \times \{k\}\}$, *where* $\omega = e^{\pi i/2m}$.

Type 3 (i). $C_m \times E_O$.

Type 3 (ii). $\{C_m \times E_O^{(0)}\} \cup \{\omega C_m \times E_O^{(1)}\} \cup \{\omega^2 C_m \times E_O^{(2)}\}$, *where* $\omega = e^{2\pi i/3m}$.

Type 4. $C_m \times \{E_O \cup V_O\}$.

Type 5. $C_m \times E_I$.

Since $\tilde{L}$ has many components, it seems complicated to draw $\tilde{L}$ in general; however, if L is of type 1, then we can do that by noting the following fact: Each component $\tilde{K}$ of $\tilde{L}$ intersects S^1 and $S^1 j$ at $\{\pm\omega_1^r\}$ and $\{\pm\omega_2^r j\}$ respectively, where r is an integer. Figure 5.2 gives the geodesic link obtained from the 2-bridge knot of type $(5, 2)$.

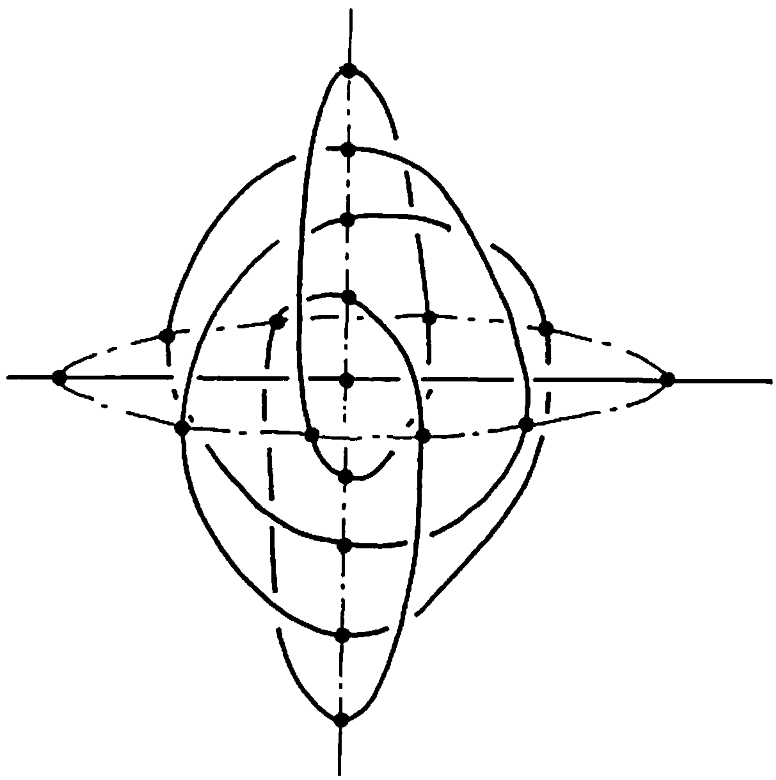


Fig. 5.2

6. The symmetry groups of spherical Montesinos links

In this section, we assume the orbifold uniformization theorem announced by Thurston [T2] and prove the following.

THEOREM 6.1. *Let L be a spherical Montesinos link. Then the symmetry group $Sym(S^3, L)$ of L is isomorphic to $\pi_0(IsomO(L))$.*

PROOF. If the exterior $E(L)$ of L is a Seifert fibred space (cf. Proposition 3.6), then we can calculate $Sym(S^3, L)$ by using [Jo, Proposition 25.3], and obtain the desired result. So we assume $E(L)$ is not a Seifert fibred space. Since $E(L)$ does not contain an essential torus by [BnS2, Theorem 9.12], L is a hyperbolic link by the uniformization theorem of Thurston (cf. [MB]); that is, $intE(L)$ admits a complete hyperbolic structure of finite volume. By [Wa], $Sym(S^3, L)$ is isomorphic to the subgroup of $Out(\pi_1(E(L)))$ consisting of those elements which preserve the meridians. Since $Out(\pi_1(E(L))) \cong Isom(intE(L))$ by Mostow's rigidity theorem (cf. [T1]), $Sym(S^3, L)$ is realized as a finite group action $\mathcal{S}$ on (S^3, L). Let M be the double cover of S^3 branched over L, and let $\tilde{\mathcal{S}}$ be the group consisting of the lifts of elements of $\mathcal{S}$ to M. Then the covering involution τ generates a normal subgroup of $\tilde{\mathcal{S}}$ such that $\tilde{\mathcal{S}}/\langle\tau\rangle \cong \mathcal{S}$. Since τ has a 1-dimensional fixed point set, it follows from the orbifold uniformization theorem [T2], that $\tilde{\mathcal{S}}$ is geometric. Thus we may assume $\tilde{\mathcal{S}} \subset IsomM$. Then we have $\tilde{\mathcal{S}}/\langle\tau\rangle \subset IsomO(L)$. Since $\mathcal{S}$ realizes the full symmetry group, we have $\tilde{\mathcal{S}}/\langle\tau\rangle = IsomO(L)$. This completes the proof.

Together with Theorem 4.1, this determines the symmetry groups of spherical Montesinos links. In particular, it follows that Theorem 1.3 of [BiZ1] also holds for spherical Montesinos links with 3 branches. That is;

THEOREM 6.2. *For a spherical Montesinos link $L = L(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$, there is an exact sequence:*

$$1 \longrightarrow Z_2 \longrightarrow Sym(S^3, L) \longrightarrow D_+(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \longrightarrow 1.$$

Here $D_+(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ denotes the group of those (dihedral) permutations of the components of the vector $v = (\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \in (Q/Z)^3$ which preserve v.

APPLICATION 6.3. The knots 8_{10} and 8_{20} in the table of [Ro] are the spherical Montesinos links $L(0; (2,1), (3,1), (3,2))$ and $L(1; (2,1), (3,1), (3,2))$ respectively. Thus their symmetry groups are isomorphic to Z_2; the generators are realized by strong inversions of the knots. Thus it follows that 8_{10} and 8_{20} have no free periods. This had been announced in [Sa] (cf. [KS]).

Finally, we note that the symmetry groups of 2-bridge knots and links are already determined without appealing to [T2]. For 2-bridge knots, it is reported that Conway calculated the outer automorphism groups of their knot groups (see [GLM]). In [BnS2], Bonahon and Siebenmann determined the symmetry groups of 2-bridge knots and links using the result of Schubert [Sb] that the 2-bridge decompositions of them are unique up to isotopy. In it, the symmetry groups of all algebraic links except Montesinos links with 3 branches are also determined.

References

[BiZ1] M. Boileau and B. Zimmermann; *Symmetries of nonelliptic Montesinos links*, Math. Ann. **277** (1987), 563–584.

[BiZ2] M. Boileau and B. Zimmermann; *The π-orbifold group of a link*, Math. Z. **200** (1989), 187–208.

[BnS1] F. Bonahon and L. Siebenmann; *The classification of Seifert fibred 3-orbifolds*, London Math. Soc. Lect. Note Series **95**, 19–85.

[BnS2] F. Bonahon and L. Siebenmann; *Geometric splittings of knots, and Conway's algebraic knots*, preprint.

[Br] G. Bredon; Introduction to compact transformation groups, Academic Press, (1972).

[Bu1] G. Burde; *Verschringungsinvarianten von Knoten und Verkettungen mit zwei Brücken*, Math. Z. **145** (1975), 235–242.

[Bu2] G. Burde; *Links covering knots with two bridges*, Kobe J. Math. **5** (1988), 209–220.

[BuM] G. Burde and K. Murasugi; *Links and Seifert fiber space*, Duke Math. J. **37** (1970), 89–93.

[BuZ] G. Burde and H. Zieschang; Knots, De Gruyter Studies in Mathematics 5, de Gruyter, Berlin, Ney York, (1985).

[C] P. E. Conner; *Transformation groups on $K(\pi, 1)$ II*, Mich. Math. J. **6** (1959), 413–417.

[D1] W. D. Dunbar; *Fibered orbifolds and crystallographic groups*, Thesis, Princeton Univ., (1981).

[D2] W. D. Dunbar; *Geometric orbifolds*, Rev. Mat, Univ. Complutense 1 (1988), 67–99.

[GW] H. Gluck and F. W. Warner; *Great circle fibrations of the three sphere*, Duke Math. J. **50** (1983), 107–132.

[GLM] C. McA. Gordon, R. A. Litherland, and K. Murasugi; *Signatures of covering links*, Canad. J. Math. **33** (1981), 381–394.

[HK] R. Hartley and A. Kawauchi; *Polynomials of amphicheiral knots*, Math. Ann. **243** (1989), 63–70.

[H] H. Hopf; *Zum Clifford-Kleinschen Raumproblem*, Math. Ann. **95** (1925), 313–339.

[J] K. Johannson; Homotopy equivalence of 3-manifolds with boundaries, Lect. Notes in Math. 761, Springer Verlag, (1979).

[KS] K. Kodama and M. Sakuma; *Symmetry groups of prime knots up to 10 crossings*, preprint.

[M1] J. M. Montesinos; *Variedades de Seifert que son recubridores ciclios ramificedos de dos hojas*, Bol. Soc. Mat. Mexicana **18** (1973), 1–32.

[M2] J. M. Montesinos; Classical Tessellations and Three-manifolds, Universitext, Springer Verlag, (1987).

[MB] J. Morgan and H. Bass; The Smith conjecture, Academic Press, (1984).

[NR] W. D. Neumann and F. Raymond; *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*, Lect. Notes in Math. **664** (1977), 161–195, Springer Verlag.

[O] P. Orlick; Seifert manifolds, Lect. Notes in Math. 291, Springer Verlag, (1972).

[RV] F. Raymond and A. T. Vasquez ; *3-manifolds whose universal covers are Lie groups*, Topolgy and its Application 12 (1981), 161–179.

[Ro] D. Rolfsen; Knots and links, Math. Lect. Note Series 7, Publish or Perish Inc., (1976).

[Sa] M. Sakuma; *Non-free-periodicity of amphicheiral hyperbolic knots*, Homotopy Theory and related topics, Adv. Stud. in Math. 9 (1986), Kinokuniya and North-Holland.

[Sr] O. Schreier; *Über die Gruppen* $A^a B^b = 1$, Abh. Math. Sem. Univ. Hamburg 3 (1924), 167–169.

[Sb] H. Schubert; *Knoten mit zwei Brüken*, Math. Z. 65 (1956), 133–170.

[St] P. Scott; *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.

[TS] W. Threlfall and H. Seifert; *Topologische Untersuchung der Discontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raums II*, Math. Ann. 107 (1932), 543–586.

[T1] W. Thurston; *The geometry and topology of 3-manifolds*, mimeographed note.

[T2] W. Thurston ; *Three manifolds with symmetry*, preliminary report.

[Wa] F. Waldhausen; *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. 87 (1968), 56–88.

[Wo] J. A. Wolf; Spaces of constant curvature, McGraw-Hill, New York, (1967).

[Z] H. Zieschang; *Classification of Montesinos knots*, Topology Proc. Leningrad 1982, Lect Notes in Math. 1060, (1984), 378–389, Springer Verlag.

Department of Mathematics,
College of General Education,
Osaka University,
Toyonaka, Osaka, 560, Japan