REALIZATION OF THE SYMMETRY GROUPS OF LINKS

Dedicated to Professor Kunio Murasugi on his sixtieth birthday

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In this paper, we consider the following problem:

Problem. For a link L in S^3, can we *see* all *symmetries* of L simultaneously?

The meaning of this problem varies according to the interpretation of the word "*see*" and "*symmetry*". We interpret the term "*symmetry*" in two ways:

(1) A rigid symmetry of L; that is, a finite subgroup of $Diff(S^3,L)$.
(2) The symmetry group $Sym(S^3,L) = \pi_0 Diff(S^3,L)$.

There is a close relation between these two interpretations of "symmetry". In fact, if L is an unsplittable link whose complement does not admit a circle action, then we almost have the following one to one correspondence (cf. [BZ2,Theorem 2.1]):

(0.1) {Finite subgroups of $Diff(S^3,L)$}/conjugation
$\longleftrightarrow$ {Finite subgroups of $Sym(S^3,L)$}/conjugation

In fact Borel's Theorem (cf.[CR]) and the result of [FY] say that the restriction of the natural map $\pi_0 : Diff(S^3,L) \longrightarrow Sym(S^3,L)$ to a finite subgroup of $Diff(S^3,L)$ is injective; and the results of [Z1,Z2] say that each finite subgroup of $Sym(S^3,L)$ has a unique "geometric" realization. Moreover, [Th3] asserts that any orientation preserving finite group action on (S^3,L) is "geometric".

The term "*see*" also is interpreted in two ways:

(1) *See* in the standard 3-sphere; that is, S^3 equipped with the standard Riemannian metric.
(2) *See* in the smooth 3-sphere forgetting the metric.

Thus the precise meanings of the problem are as follows:

(1) Is there a representative $L^* \subset S^3$ of the link type $[L]$ determined by $L \subset S^3$, such that any rigid symmetry (up to conjugation)

is realized as a group of isometries of the standard 3-sphere S^3 respecting L^*?

(2) Does the symmetry group of L have an isometric realization? That is, is there a representative $L^* \subset S^3$ of the link type $[L]$, for which there is a homomorphism $\eta : Sym(S^3, L^*) \longrightarrow Isom(S^3, L^*)$ such that the following diagram is commutative?

$$\begin{array}{ccc} Diff(S^3, L^*) & \xrightarrow{\pi_0} & Sym(S^3, L^*) = Sym(S^3, L) \\ \cup & \swarrow{\scriptstyle \eta} & \\ Isom(S^3, L^*) & & \end{array}$$

Here $Isom(S^3, L^*)$ is the group of isometries of S^3 respecting L^*.

(3) Does the symmetry group of L have a smooth realization? That is, does the natural homomorphism $\pi_0 : Diff(S^3, L) \longrightarrow Sym(S^3, L)$ have a right inverse?

An affirmative answer to Problem (2) implies an affirmative answer to (3) and an almost affirmative answer to (1) because of the correspondence (0.1). For some classes of links including 2-bridge knots and nonelliptic Montesinos links, Problem (2) is proved to be affirmative (see [BS, BZ1]). However, the following observation implies that Problem (2) is negative in general.

Observation 0.2. If $Sym(S^3, L)$ has an isometric realization, then it is a finite group.

This follows from the fact that $Isom(S^3, L)$ is a closed subgroup of the compact Lie group $Isom(S^3)$, and hence it has only finitely many connected components. If L is hyperbolic, then $Sym(S^3, L)$ is finite by Mostow's rigidity theorem; however, if L is non-hyperbolic, then $Sym(S^3, L)$ is infinite in general (see Section 1). In Section 2, we calculate the symmetry group of a certain knot K, and show that for this knot Problems (1) and (2) are negative, but (3) is positive (Proposition 2.7). So we cannot see all rigid symmetries of K simultaneously in the standard Riemannian 3-sphere, but if we forget the metric, we can do so. In Section 3, we partially generalize the above result and show that Problem (3) is positive for a certain class of links with infinite symmetry groups (Theorem 3.1). In Section 4, we determine all unsplittable non-hyperbolic links whose symmetry groups are finite, and show that their symmetry groups have isometric realizations (Theorem 4.1).

1. Dehn twists along an essential torus.

Let T be a separating incompressible torus in a Haken manifold M. Then M is decomposed into a union $M_0 \cup T\times[0,1] \cup M_1$ where $T\times[0,1]$ is identified with the regular neighbourhood of T and $M_i \cap T\times[0,1] = T\times\{i\}$ $(i = 0,1)$. We identify the universal cover of T with $\mathbb{R}^2$ and $\pi_1(T)$ with the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$; thus T is identified with $\mathbb{R}^2/\mathbb{Z}^2$. For each element α of $\pi_1(T)$, let D_α be an element of $Diff(M)$ defined as follows:

$$\begin{cases} D_\alpha|_{M_i} = id_{M_i} & (i = 0,1), \\ D_\alpha([\vec{x}],t) = ([\vec{x}+\varphi(t)\vec{\alpha}],t) & \text{for each } ([\vec{x}],t) \in (\mathbb{R}^2/\mathbb{Z}^2)\times[0,1]. \end{cases}$$

Here $\vec{\alpha}$ is an element of $\mathbb{Z}^2\subset\mathbb{R}^2$ corresponding to α, and $[\vec{x}]$ denotes the point of $\mathbb{R}^2/\mathbb{Z}^2$ determined by $\vec{x} \in \mathbb{R}^2$. φ is a smooth function on $\mathbb{R}$ such that $\varphi(-\infty,0] = 0$, $\varphi[1,\infty) = 1$, and $\varphi|_{[0,1]}$ is increasing. We call D_α the Dehn twist along T in the direction α. Let $\mathcal{D}$ be the subgroup of $\pi_0 Diff(M)$ generated by Dehn twists along T. Then we obtain the following (cf.[Jo,Si]).

Lemma 1.1. $\mathcal{D} \cong \pi_1(T)/(Z(\pi_1(M_0))+Z(\pi_1(M_1)))$, where $Z(\ )$ denotes the center of a group.

Proof. Pick a base point for π_1 on $T\times 0$, and identity $\pi_1(M)$ with the amalgamated free product $\pi_1(M_0) \underset{\pi_1(T)}{*} \pi_1(M_1)$. Then, for each $\alpha \in \pi_1(T)$, we have

$$(D_\alpha)_*(x) = \begin{cases} x & \text{if } x \in \pi_1(M_0) \quad \text{(i)} \\ \alpha x\alpha^{-1} & \text{if } x \in \pi_1(M_1) \quad \text{(ii)} \end{cases}$$

Assume that D_α is isotopic to id. Then the isomorphism $(D_\alpha)_*$ of $\pi_1(M)$ induced by D_α is an inner automorphism by an element, say β, of $\pi_1(M)$. Then by (i), β is an element of the centralizer of $\pi_1(M_0)$ in $\pi_1(M)$; which is equal to $Z(\pi_1(M_0))$. Similary, by (ii), $\beta^{-1}\alpha$ is an element of $Z(\pi_1(M_1))$. Hence, by [W, Corollary 7.5], D_α is isotopic to id, iff there is an element β of $Z(\pi_1(M_0))$, such that $\beta^{-1}\alpha \in Z(\pi_1(M_1))$. This condition is equivalent to the condition that α is an element of $Z(\pi_1(M_0))\cdot Z(\pi_1(M_1))$. Thus we obtain the desired result.

Let $\mathcal{H}$ be the subgroup of $\pi_0 Diff(M)$ generated by diffeomorphisms of M which preseve M_i $(i=0,1)$, and let Δ be the subgroup of $\pi_0 Diff(M_0,T)\times \pi_0 Diff(M_1,T)$ consisting of all elements $([f_0],[f_1])$ such that $f_0|_T$ is isotopic to $f_1|_T$. Then we have the following exact sequence:

Lemma 1.2. $1 \longrightarrow \mathcal{D} \longrightarrow \mathcal{H} \longrightarrow \Delta \longrightarrow 1.$

Proof. This follows from the following two facts: (1) If two diffeomorphisms which preserve T are isotopic, then there is an isotopy between them which preserves T (see [W, Proof of Theorem 7.1]). (2) Any diffeomorphism of $T\times[0,1]$, which is the identity on the boundary, is isotopic to a Dehn twist relative to the boundary.

2. An example.

Let K_1 be a 2-bridge knot of type (p,q) with $q^2 \not\equiv \pm 1$ mod. p, and K be an (untwisted) double of K_1. Then K is contained in a solid torus V with core K_1, and the exterior $E(K) = S^3-\mathring{N}(K)$ of K is decomposed into a union $E(K) = M_0 \cup T\times[0,1] \cup M_1$, where $M_0 = V-\mathring{N}(K)$ and $M_1 = E(K_1) = S^3-\mathring{N}(V)$. Here $N(\ )$ denotes a regular neighbourhood; and for each $i = 0, 1$, the intersection $M_i \cap T\times[0,1]$ is equal to $T\times\{i\}$, which we denote by T_i. Let l and m be the logitude and the meridian curves of K_1 lying on T. Note that M_i $(i=0,1)$ are hyperbolic manifolds (see [Th2]); so we can identify them with the complements of open cusps of the corresponding complete hyperbolic manifolds. Thus each T_i admits a Euclidian structure (see [Th2]); that is, l and m of $\pi_1(T_i)$ $(= \pi_1(T))$ correspond to translations of the Euclidian plane $\mathbb{R}^2$ by certain vectors $\vec{l}_i$ and $\vec{m}_i$ respectively, and T_i is identified with the quotient $\mathbb{R}^2/\langle \vec{l}_i, \vec{m}_i\rangle$.

Lemma 2.1. (See Figures 2.1 and 2.2.)

(1) $\pi_0 Diff(M_0,T_0) \cong Isom(M_0,T_0) = \langle f | f^2=1\rangle \oplus \langle h | h^2=1\rangle$, and the restriction of them to T_0 are given by $f[\vec{x}] = [\vec{x} + \frac{1}{2}\vec{m}_0]$, $h[\vec{x}] = [-\vec{x}]$.

(2) $\pi_0 Diff(M_1,T_1) \cong Isom(M_1,T_1) = \langle g | g^2 = 1\rangle \oplus \langle k | k^2 = 1\rangle$, and the restriction of them to T_1 are given by $g[\vec{x}] = [\vec{x} + \frac{1}{2}\vec{l}_1]$, $k[\vec{x}] = [-\vec{x}]$.

Proof. These follow from Conway's calculation of the outer-automorphism groups of the 2-bridge knot groups (see [GLM]) and Property P for 2-bridge knots ([Ta]) (or the recent result of [GL]).

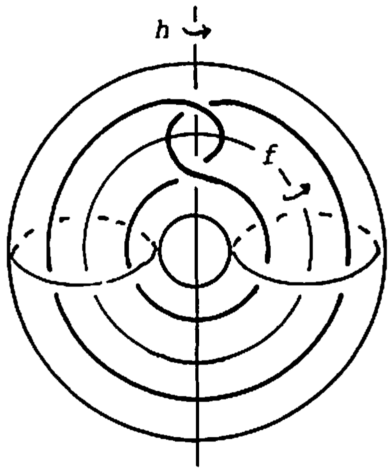


Fig. 2.1

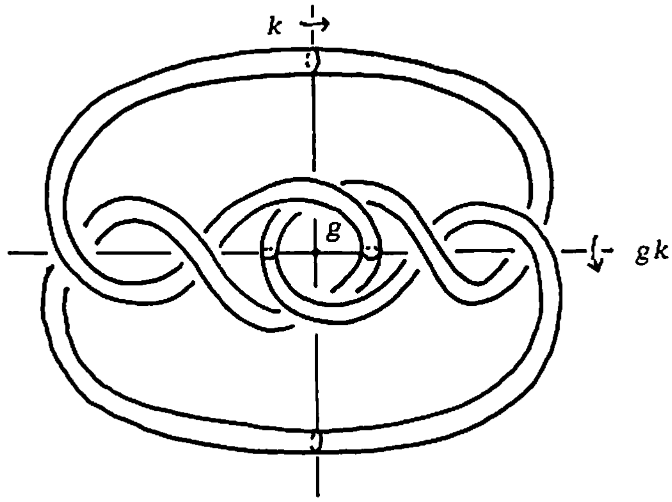


Fig. 2.2

Thus, the group Δ defined in Section 1 is given by

$$\Delta = \langle (f, id), (id, g), (h, k) \rangle \cong (\mathbb{Z}_2)^3.$$

Since T is characteristic and $Z(\pi_1(M_i)) \cong 1$ $(i = 0,1)$, we obtain the following exact sequence by Lemmas 1.1 and 1.2.

$$(2.2) \quad 1 \longrightarrow \underset{\displaystyle \pi_1(T)}{\overset{\mathcal{D}}{\wr\Vert}} \longrightarrow \pi_0 Diff(E(K)) \longrightarrow \underset{\displaystyle (\mathbb{Z}_2)^3}{\overset{\Delta}{\wr\Vert}} \longrightarrow 1$$

The precise presentation of this group is obtained from its smooth realization. To see this, note that each T_i has a collar neighbourhood C_i in M_i on which $Isom(M_i, T_i)$ acts as a product action. Choose any Euclidian torus $T = \mathbb{R}^2/\langle \vec{l}, \vec{m} \rangle$. Then the linear map of $\mathbb{R}^2$ which sends $\vec{l}$ and $\vec{m}$ to $\vec{l}_i$ and $\vec{m}_i$ respectively induces an affine equivalence between two Euclidian tori T and T_i $(i = 0,1)$. By the uniqueness of smooth structures on 3-manifolds (cf.[L]), we may assume that the smooth structure on $E(K)$ is obtained from those on M_0, M_1 and $T\times[0,1]$, by identifying $T_i \subset M_i$ with $T\times\{i\} \subset T\times[0,1]$ through the above affine equivalence and by using the equivariant collar neighbourhood C_i of T_i and the product structure of $T\times[0,1]$ (cf. [H, p.184]). For a vector $\vec{\alpha} \in \mathbb{R}^2$, let $D(\vec{\alpha})$, $S(\vec{\alpha})$ and R be the diffeomorphisms of $T\times[0,1]$ defined as follows:

$$D(\vec{\alpha})([\vec{x}], t) = ([\vec{x} + \varphi(t)\vec{\alpha}], t),$$
$$S(\vec{\alpha})([\vec{x}], t) = ([\vec{x} + \vec{\alpha}], t),$$
$$R([\vec{x}], t) = ([-\vec{x}], t).$$

Let F, G, and H be maps of $E(K)$ defined as follows. (Note that they are elements of $Diff(E(K))$ by the preceding remark on the smooth structure on $E(K)$.)

$F|_{M_0} = f,\ F|_{M_1} = id,\ F|_{T\times[0,1]} = D(-\vec{m}/2)\circ S(\vec{m}/2),$
$G|_{M_0} = id,\ G|_{M_1} = g,\ G|_{T\times[0,1]} = D(\vec{l}/2),$
$H|_{M_0} = h,\ H|_{M_1} = k,\ H|_{T\times[0,1]} = R.$

Put $L = D_l$ $(= id_{M_0} \cup D(\vec{l}) \cup id_{M_1})$ and $M = D_m$ $(= id_{M_0} \cup D(\vec{m}) \cup id_{M_1})$. Then we have the following equality in $Diff(E(K))$:

(2.3) $F^2 = M^{-1},\ G^2 = L,\ H^2 = 1,$
$[F,G] = 1,\ [H,F] = M,\ [H,G] = L^{-1}.$

Here $[X,Y]$ denotes the commutator $XYX^{-1}Y^{-1}$ of X and Y. Moreover, L and M generate a free abelian subgroup of rank 2 in $Diff(E(K))$, which is invariant by the inner-automorphisms induced by F,G, and H. In fact, we have

(2.4) $[L,M] = 1,\ \iota_F = id,\ \iota_G = id,\ \iota_H = -id.$

Here, ι_Y denotes an inner automorphism $X \to YXY^{-1}$ on $\langle L,M\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$ induced by Y ($= F,G$ or H). Thus the subgroup $\mathcal{G}$ of $Diff(E(K))$ generated by $\{L,M,F,G,H\}$ satisfies the following exact sequence;

$$1 \longrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\wr\|}{\langle L,M\rangle}} \longrightarrow \mathcal{G} \longrightarrow (\mathbb{Z}_2)^3 \longrightarrow 1.$$

Hence the natural homomorphism from $\mathcal{G}$ to $\pi_0 Diff(E(K))$ is an isomorphism by (2.2). Thus $\mathcal{G}$ gives a realization of $\pi_0 Diff(E(K))$, and its presentation is given by (2.3) and (2.4). Clearly $\mathcal{G}$ extends to a group of diffeomorphisms of the pair (S^3,K), and hence $Sym(S^3,K) \cong \pi_0 Diff(E(K))$ has a smooth realization. From the group presentation, we can determine all torsion elements of $Sym(S^3,K)$. In fact we have the followings:

(2.5) The set of all torsion elements of $Sym(S^3,K)$ is the disjoint union $\mathcal{D}H \cup \mathcal{D}GH \cup \mathcal{D}FH \cup \mathcal{D}FGH$.

(2.6) Each element of $\mathcal{D}H$ (resp. $\mathcal{D}GH$, $\mathcal{D}FH$, and $\mathcal{D}FGH$) is conjugate to H (resp. GH, FH, and FGH). Moreover, H, GH, FH, and FGH are not conjugate to each other.

In conclusion, any finite subgroup of $Sym(S^3,K)$ is conjugate to precisely one of the subgroups $\langle H\rangle$, $\langle GH\rangle$, $\langle FH\rangle$, and $\langle FGH\rangle$. Further, each of them has an isometric realizations as illustrated in Figure 2.3. Here H and GH give strong inversions of K, and FH and FGH realizes the cyclic period 2 of K. No two of them can be seen simultaneously in the standard Riemannian 3-sphere. In fact, any two nonconjugate torsion elements of $Sym(S^3,K)$ generate infinite subgroup, which has no isometric realization by Observation 0.2. We summarize our results.

Proposition 2.7. (1) $Sym(S^3,K)$ has a smooth realization; but it does not admit an isometric realization.

(2) $Sym(S^3,K)$ has precisely four finite subgroups up to conjugation, and each of them has an isometric realization. But no two of them have simultaneous isometric realizations.

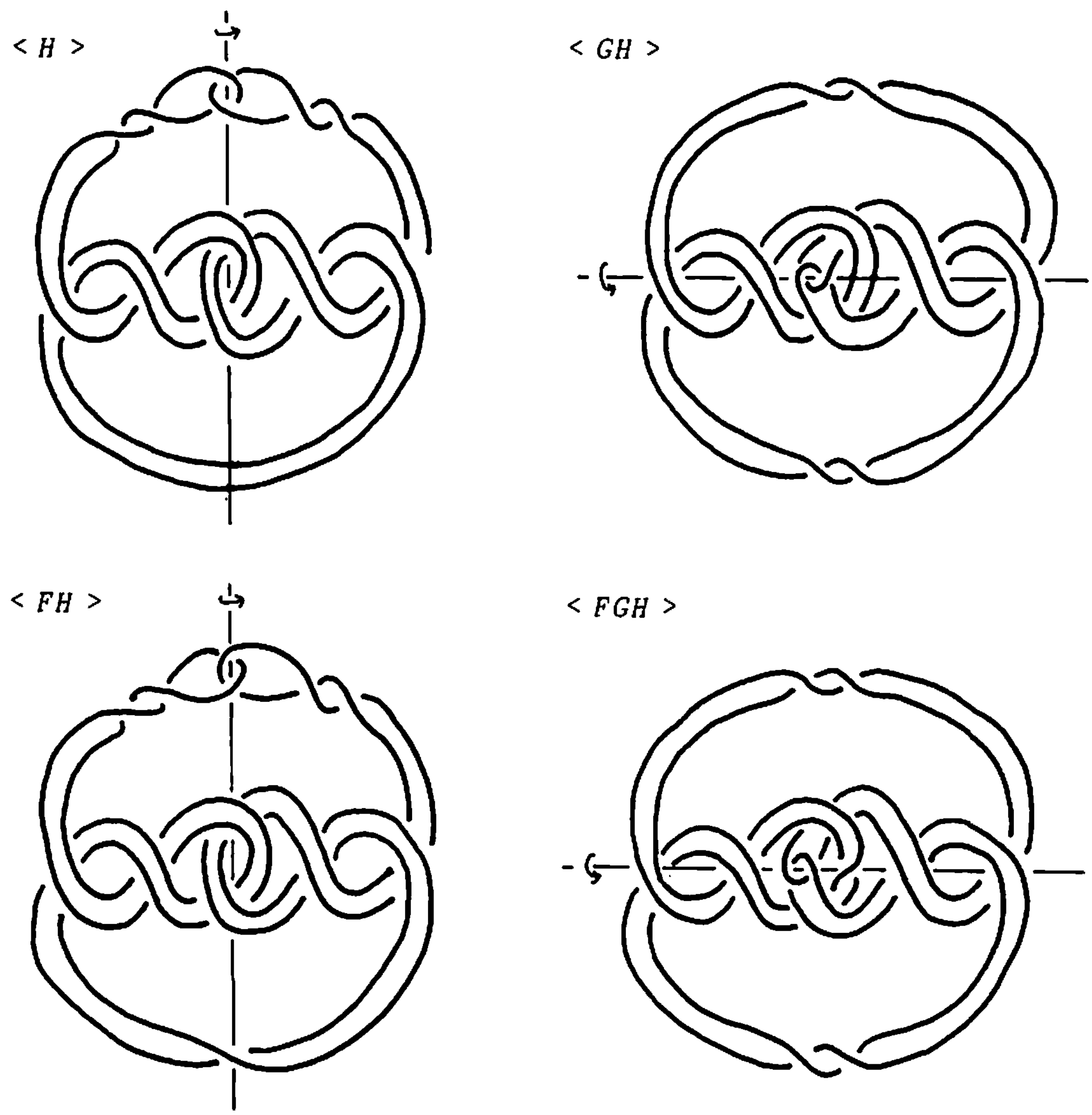


Fig. 2.3

Remark 2.8. (1) For any integer n, there is a knot whose symmetry group has more than n number of subgroups of order 2 up to conjugation [Sa, Claim 2]. The knots constructed in it also have the property (1) of Proposition 2.7 by virtue of the following Theorem 3.1.

(2) In [Sa, Theorem 1], we gave a sufficient condition for a knot K to have the "universal" rigid symmetry; that is, a finite subgroup G^* of $Sym(S^3,K)$ such that any finite subgroup of $Sym(S^3,K)$ is conjugte to a subgroup of G^*.

3. Manifolds of split hyperbolic type.

A compact 3-manifold M is said to be of split hyperbolic type, if there is a collection $\mathcal{T}$ of disjoint tori in $intM$ such that $intM-\mathcal{T}$ is a complete hyperbolic manifold of finite volume (cf.[Si]). We prove the following theorem.

Theorem 3.1. (1) Let M be a compact 3-manifold of split hyperbolic type. Then $\pi_0 Diff(M)$ has a smooth realization.

(2) Let L be a link whose exterior is of split hyperbolic type. Then $Sym(S^3,L)$ has a smooth realization.

Proof. (1) As in Section 2, M is obtained from a disjoint union of compact manifolds $\hat{M} \cup \mathcal{T}\times[0,1]$ by glueing their boundaries. Here $\hat{M}$ is obtained from a (possibly disconnected) complete hyperbolic manifold of finite volume by delating open cusps which are invariant by the action of the isometry group. Note that the components of $\partial\hat{M}$ admit Euclidian structures which are invariant by the action of $Isom(\hat{M})$. On the other hand $\mathcal{T}$ is a disjoint union of copies (T_i) of a Euclidian torus $T = \mathbb{R}^2/\mathbb{Z}^2$, and each boundary of $\mathcal{T}\times[0,1]$ is identified with a component of $\partial\hat{M}$ via an affine transformation. Let $\hat{\Delta}$ be the subgroup of $Isom(\hat{M})$ consisting of those isometries which extend to diffeomorphisms of M. Then by Mostow's rigidity theorem and an argument similar to that of Section 1, we obtain the following exact sequence:

$$1 \longrightarrow \mathcal{D} \longrightarrow \pi_0 Diff(M) \longrightarrow \hat{\Delta} \longrightarrow 1.$$

Here $\mathcal{D}$ is the subgroup generated by Dehn twists along $\mathcal{T}$, and is isomorphic to the free abelian group of rank $2|\mathcal{T}|$ (see [Jo, p.188]). ($|\mathcal{T}|$ denotes the number of connected components of $\mathcal{T}$.) Let $\mathcal{A}$ be the subgroup of $Diff(T\times[0,1])$ which is generated by the following three types of diffeomorphisms:

(1) (affine automorphism)$\times id$.

(2) The map $D(\vec{\alpha})$ ($\vec{\alpha} \in \mathbb{R}^2$) defined in Section 2. Here we assume that the smooth function φ used to define $D(\vec{\alpha})$ satisfies $\varphi(1-t) = 1-\varphi(t)$ for any $t\in\mathbb{R}$.

(3) The diffeomorphism: $([\vec{x}],t) \longrightarrow ([\vec{x}],1-t)$.

Let $\mathcal{G}$ be the subset of $Diff(M)$ consisting of all elements (F) which satisfy the following two conditions:

(1) $F|_{\hat{M}}$ is an element of $\hat{\Delta}$.

(2) Since all components of $\mathcal{T}$ are copies of the Euclidian torus

T, $F|_{T_i\times[0,1]}$ determines an element of $Diff(T\times[0,1])$ for each component T_i of $\mathcal{T}$. We require that this elememt belongs to $\mathcal{A}$.

Clearly $\mathcal{G}$ forms a subgroup of $Diff(M)$. We claim that $\mathcal{G}$ gives a smooth realization of $\pi_0 Diff(M)$; that is, the natural map $\mathcal{G} \longrightarrow \pi_0 Diff(M)$ is an isomorphism. We first prove the surjectivity. It is clear that any Dehn twist along $\mathcal{T}$ is realized by an element of $\mathcal{G}$. So we have only to show that the map $\mathcal{G} \longrightarrow \pi_0 Diff(M) \longrightarrow \hat{\Delta}$ is surjective. For an element f of $\hat{\Delta}$, there is a diffeomorphism $\bar{f}$ of M, such that $\bar{f}|_{\hat{M}} = f$. For each $T_i \subset \mathcal{T}$, let f_i be a diffeomorphism of $T\times[0,1]$ determined by $\bar{f}|_{T_i\times[0,1]}$. Since $f \in Isom(\hat{M})$, the restrictions of f_i to the boundary components of $T\times[0,1]$ are affine automorphisms. That is, there are A_0, $A_1 \in GL(2,\mathbb{Z})$ and $\vec{\alpha}_0$, $\vec{\alpha}_1 \in \mathbb{R}^2$, such that

$$f_i([\vec{x}],0) = ([A_0(\vec{x})+\vec{\alpha}_0],\varepsilon_0),$$
$$f_i([\vec{x}],1) = ([A_1(\vec{x})+\vec{\alpha}_1],\varepsilon_1). \quad \text{Here} \quad \{\varepsilon_0,\varepsilon_1\} = \{0,1\}.$$

Since $f_i|_{T\times 0}$ and $f_i|_{T\times 1}$ are isotopic, we have $A_0 = A_1 \in GL(2,\mathbb{Z})$. Let F_i be a diffeomorphism of $T\times[0,1]$ defined as follows:

$$F_i([\vec{x}],t) = ([A_0(\vec{x})+\varphi(1-t)\vec{\alpha}_0+\varphi(t)\vec{\alpha}_1],\varepsilon_1 t+\varepsilon_0(1-t))$$

Then F_i is an element of $\mathcal{A}$ and $F_i|_{T\times\{0,1\}} = f_i|_{T\times\{0,1\}}$. We use the same symbol F_i to denote the map $T_i\times[0,1] \to \bar{f}(T_i\times[0,1])$ $(= T_j\times[0,1]$ for some j) determined by $F_i \in \mathcal{A}$. Then F_i and $\bar{f}$ are identical on $\partial(T_i\times[0,1])$. Thus $f = \bar{f}|_{\hat{M}}$ together with $\{F_i\}$ determines a diffeomorphism F of M. Clearly F is an element of $\mathcal{G}$ and is mapped to f by the natural map $\mathcal{G} \longrightarrow \hat{\Delta}$. Hence the homomorphism $\mathcal{G} \longrightarrow \pi_0 Diff(M)$ is surjective. Finally we show the injectivity. Suppose that an element F of $\mathcal{G}$ is isotopic to the identity. Then, by Mostow's rigidity theorem, $F|_{\hat{M}} = id|_{\hat{M}}$. Moreover for each $T_i \subset \mathcal{T}$, $F|_{T_i\times[0,1]}$ is isotopic to the identity relative to the boundary. Note that, if an element of $\mathcal{A}$ is isotopic to the identity relative to the boundary, then it is the identity. [Here we use the condition $\varphi(1-t) = 1-\varphi(t)$.] Therefore, $F|_{T_i\times[0,1]} = id$, and hence $F = id$. This completes the proof of Theorem 3.1 (1).

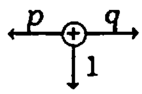
(2) Since $Sym(S^3,L)$ is a subgroup of $\pi_0 Diff(E(L))$, it acts on $E(L)$ by the above proof; and clearly, this action extends to a smooth action on (S^3,L).

4. Unsplittable non-hyperbolic links with finite symmetry groups.

In this section we determine the links with the above property. Let L be such a link, and consider the torus decomposition (see [Ja,Jo,Th1]) of $E(L)$. Then, by Lemmas 1.1 and 1.2, one of the following holds:

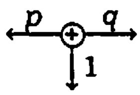
(1) $E(L)$ is a Seifert fibered space without essential tori.

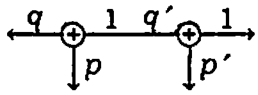
(2) $E(L)$ is decomposed into a union of two Seifert fibered spaces without essential tori.

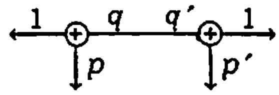
If L satiesfies (1), then L is a sublink of the Seifert link represented by the diagram 

. Here we use the notation of [EN], and p and q are relatively prime integers. $Sym(S^3,L)$ is a certain subgroup of $\pi_0 Diff(E(L))$, which can be calculated by [Jo, Proposition 25.3]. If L satisfies (2), then L is obtained from two links satisfying (1) by a splicing operation (see [EN,Proposition 2.1]). Its symmetry group can be calculated by using the results of Section 1.

Theorem 4.1. Let L be an unsplittalble non-hyperbolic link whose symmetry group is finite. Then L is equivalent to a "suitable" sublink of one of the following graph links:

(I) 

(II) 

(III) 

Here we use the notation of [EN], and p and q (resp. p' and q') are relatively prime integers, such that $q'-p'pq \neq 0$ in Case II, and $pp'-qq' \neq 0$ in Case III. Moreover $Sym(S^3,L)$ can be realized as a subgroup of $Isom(S^3)$.

In the following we give the symmetry group of L and a precise construction of a representative L^* of L which allows an isometric realization of the symmetry group. To do this identify S^3 (resp. S^1) with the unit sphere in $\mathbb{C}^2$ (resp. $\mathbb{C}$), and let S_1 (resp. S_2) be the great circle $S^3 \cap (\mathbb{C}\times 0)$ (resp. $S^3 \cap (0\times\mathbb{C})$). Let Ψ be an embedding of $S^1\times S^1$ into $Isom(S^3)$, given by $\Psi(\alpha_1,\alpha_2)(z_1,z_2) = (\alpha_1 z_1, \alpha_2 z_2)$. Put $\mathrm{T} = Im\Psi$, then T consists of all isometries of S^3 which preseve S_i and its orientation for $i = 1,2$. Let C_1, C_2 and R be the isometries of S^3 given by the following formulas:

$$C_1(z_1,z_2) = (\bar{z}_1,z_2),\ C_2(z_1,z_2) = (z_1,\bar{z}_2),\ R(z_1,z_2) = (z_2,z_1).$$

Then the above isometries together with T generate the subgroup of $Isom(S^3)$ consisting of all isometries which preseve $S_1 \cup S_2$ (cf. Figure 4.0).

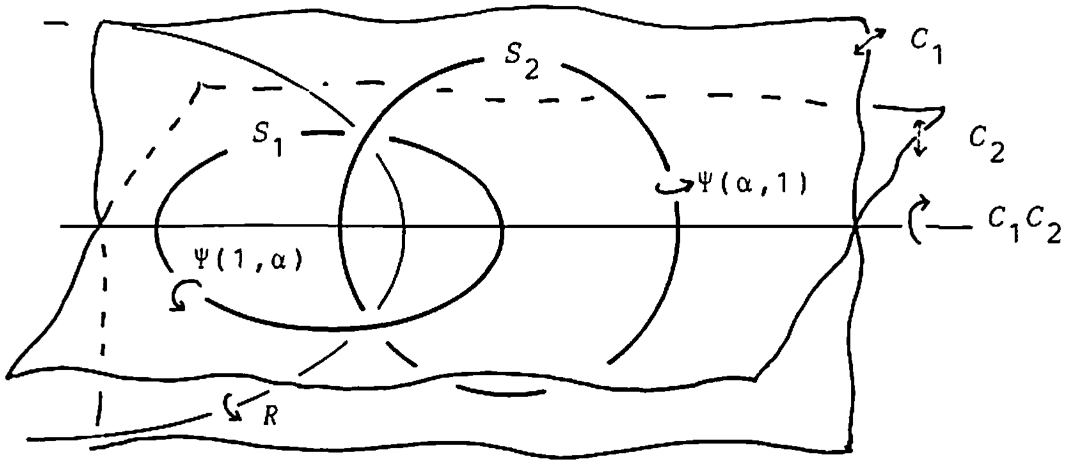


Fig. 4.0

For relatively prime integers p and q, let $\Psi_{p,q}:S^1 \to Isom(S^3)$ be the effective isometric S^1 action on S^3, defined by $\Psi_{p,q}(\alpha) = \Psi(\alpha^p,\alpha^q)$. If neither p nor q is 0, then its orbit space is S^2, and the map $\psi_{p,q}:S^3 \longrightarrow \mathbb{C}\cup\{\infty\} \cong S^2$ defined by $\psi_{p,q}(z_1,z_2) = z_2^p/z_1^q$ gives the quotient map. Moreover, $\psi_{p,q}^{-1}(0) = S_1$, $\psi_{p,q}^{-1}(\infty) = S_2$, and for any $z \in \mathbb{C}\setminus\{0\}$, $\psi_{p,q}^{-1}(z)$ forms a torus knot of type (p,q).

Case I. L is a suitable sublink of $\tilde{L} = S_1 \cup K_{p,q} \cup S_2$, where $K_{p,q} = \psi_{p,q}^{-1}(1)$. (See Figure 4.1.) Ignoring the orientations, we may assume $p \geq q \geq 0$ (cf.[EN, Theorem 8.1]), and this case is divided into the following subcases:

(I-1) $p>q>1$ and L is a sublink of $\tilde{L}$ containing $K_{p,q}$.
(I-2) $p>q=1$ and L is a sublink of $\tilde{L}$ containing $K_{p,q} \cup S_2$.
(I-3) $p=1$, $q=1$ and $L = \tilde{L}$ = 3-component Hopf link.
(I-4) $p=1$, $q=0$ and $L = \tilde{L}$ = .
(I-5) $L = S_1$ or $S_1 \cup S_2$; that is, a Hopf link with μ (≤ 2) components.

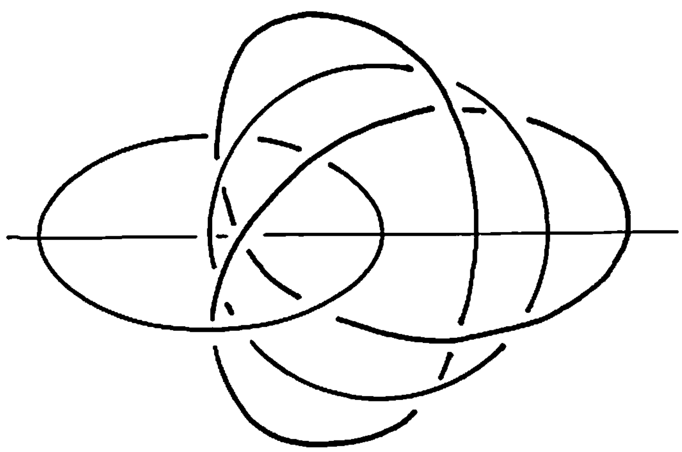


Fig. 4.1

Then $Sym(S^3,L)$ is as follows (cf.[Jo, Proposition 25.3]).

I-1	I-2	I-3	I-4	I-5
$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	D_6 ($\cong D_3 \times \mathbb{Z}_2$)	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^2$ if $\mu = 1$ D_4 if $\mu = 2$

Here D_n is the dihedral group of order $2n$. The representative $\tilde{L}^*$ of $\tilde{L}$ is given by

$S_1 \cup \psi_{p,q}^{-1}(1) \cup S_2$ in Cases I-1 (and 5),

$S_1 \cup \psi_{p,q}^{-1}(1) \cup \psi_{p,q}^{-1}(-1)$ in Cases I-2 and 4,

$\cup_{j=0}^{2} \psi_{p,q}^{-1}(\omega^j)$ with $\omega = exp(2\pi\sqrt{-1}/3)$ in Case I-3.

The representative L^* of L is given as the corresponding sublink of $\tilde{L}^*$. Then L^* allows an isometric realization of $Sym(S^3,L)$. In fact, except in Case I-5, $Isom(S^3,L^*)$ is a semi-direct product of $Im\Psi_{p,q}$ and the following subgroup, which is isomorphic to $Sym(S^3,L)$:

I-1	I-2	I-3	I-4
$\langle C_1C_2 \rangle$	$\langle \Psi(-1,1), C_1C_2 \rangle$	$\langle \Psi(\omega,\omega^2), C_1C_2, R \rangle$	$\langle \Psi(-1,1), C_1, C_2 \rangle$

In case I-5, $Sym(S^3,L^*)$ is a semi-direct product of T and $\langle C_1,C_2 \rangle \cong (\mathbb{Z}_2)^2$ or $\langle C_1,C_2,R \rangle \cong D_4$ according to $\mu = 1$ or 2.

Case II. L is a suitable sublink of $\tilde{L} = S_1 \cup K_{p,q} \cup K(p,q;p',q') \cup S_2$, where $K_{p,q} = \psi_{p,q}^{-1}(1)$ and $K(p,q;p',q')$ is a (p',q') cable of $K_{p,q}$, and $q'-p'pq \neq 0$. (See Figure 4.2.) We may assume $p>q>1$ and $p' \geq 0$. This case is divided into the following subcases:

(II-1) $p'>1$ and L contains $K(p,q;p',q')$.

(II-2) $p'=1$ and L contains $K_{p,q} \cup K(p,q;1,q')$.

(II-3) $p'=0$ and L contains $K_{p,q} \cup K(p,q;0,q')$.

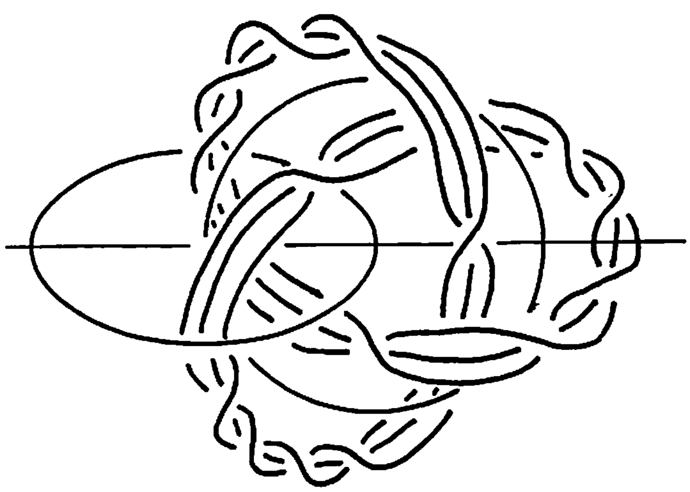


Fig. 4.2

Note that the boundary T of a tubular neighbourhood of $K_{p,q}$ whose interior contains $K(p,q;p',q')$ gives the torus decomposition of $E(L)$. By using the results of Section 1, we have

$$Sym(S^3,L) \cong \begin{cases} D_{|q'-p'pq|} & \text{in Cases II-1 and 3,} \\ D_{2|q'-p'pq|} & \text{in Case II-2.} \end{cases}$$

In fact, the subgroup $\mathcal{D}$ of $Sym(S^3,L)$ generated by Dehn twists along T is a cyclic normal subgroup of order $\left|det\begin{bmatrix} 1 & pq \\ p' & q' \end{bmatrix}\right| = |q'-p'pq|$, and the group Δ introduced in section 1 is $\mathbb{Z}_2$ [resp. $(\mathbb{Z}_2)^2$] in Cases II-1 and 3 [resp. Case II-2]. To construct the representative $\tilde{L}^*$ of $\tilde{L}$, choose an equivariant tubular neighbourhood $N(K_{p,q})$ of $K_{p,q} = \psi_{p,q}^{-1}(1)$, and a coordinate $S^1\times D^2$ for $N(K_{p,q})$, such that the restrictions of $\Psi_{p,q}(\alpha)$ $(\alpha\in S^1)$ and C_1C_2 to $N(K_{p,q})$ are given as follows:

$$\Psi_{p,q}(\alpha)(z_1,z_2) = (\alpha z_1, \alpha^{pq}z_2),$$
$$C_1C_2(z_1,z_2) = (\bar{z}_1,\bar{z}_2), \text{ for each } (z_1,z_2) \in S^1\times D^2 \cong N(K_{p,q}).$$

For ε = + or −, let $K_\varepsilon^*(p,q;p',q')$ be the circle $\{(\alpha^{p'},\varepsilon\alpha^{q'})|\alpha\in S^1\}$ on $S^1\times D^2 \cong N(K_{p,q}) \subset S^3$. Then $\tilde{L}^*$ is given by

$$S_1 \cup \psi_{p,q}^{-1}(1) \cup K_+^*(p,q;p',q') \cup S_2 \quad \text{in Cases II-1 and 3.}$$
$$S_1 \cup K_+^*(p,q;1,q') \cup K_-^*(p,q;1,q') \cup S_2 \quad \text{in Case II-2.}$$

The representative L^* of L is given as the corresponding sublink of $\tilde{L}^*$. To see this, let $\mathcal{D}^*$ be the subgroup of $Isom(S^3,L^*) \cap Im\Psi_{p,q}$ consisting of the isometries which preserve each component of L^*. Then $\mathcal{D}^* \cong \mathbb{Z}_{|q'-p'pq|}$ and is generated by $\Psi_{p,q}(\omega)$ where $\omega = exp(2\pi\sqrt{-1}/|q'-p'pq|)$, and it gives a realization of $\mathcal{D}$. In Cases II-1 and 3 (resp. II-2), $Isom(S^3,L^*)$ is generated by $\mathcal{D}^*$ and $\langle C_1C_2\rangle$ (resp. $\langle\Psi_{p,q}(\sqrt{\omega}),\ C_1C_2\rangle$) and realizes $Sym(S^3,L)$. Here $\sqrt{\omega} = exp(\pi\sqrt{-1}/|q'-p'pq|)$.

Case III. L is a suitable sublink of $\tilde{L} = S_1 \cup K_{p,q}^{(1)} \cup K_{q',p'}^{(2)} \cup S_2$, where $K_{p,q}^{(1)} = \psi_{p,q}^{-1}(1/2)$, $K_{q',p'}^{(2)} = \psi_{q',p'}^{-1}(2)$, and $pp'-qq' \neq 0$. (See Figure 4. 3). We may assume $p \geq p' \geq 0$, and this case is divided into the following five subcases:

(III-1) $p \geq p' > 1$ and L containes $K_{p,q}^{(1)} \cup K_{q',p'}^{(2)}$.

(III-2) $p > p' = 1$ and L containes $K_{p,q}^{(1)} \cup K_{q',p'}^{(2)} \cup S_2$.

(III-3) $p = p' = 1$ and $L = \tilde{L}$.

(III-4) $p > 1 > p' = 0$ and L containes $K_{p,q}^{(1)} \cup K_{q',p'}^{(2)} \cup S_2$.

(III-5) $p = p' = 0$ and $L = \tilde{L} \cong$.

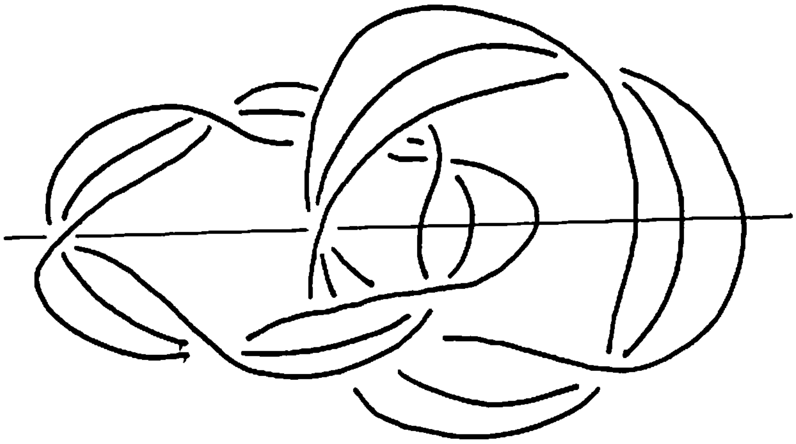


Fig. 4.3

Note that the inverse image T of the unit circle in $\mathbb{C}$ by $\psi_{p,q}$ (or $\psi_{q',p'}$) is a torus in $E(L)$ which gives the torus decomposition of $E(L)$. Let $Sym_0(S^3,L)$ be the subgroup of $Sym(S^3,L)$ which preserve each of the pieces of the torus decomposition of $E(L)$. Then we have the following:

(1) $Sym_0(S^3,L) \cong \begin{cases} D_{|pp'+qq'|} & \text{in Cases III-1 and 4,} \\ D_{2|pp'+qq'|} & \text{in cases III-2 and 5,} \\ \mathbb{Z}_2 \times D_{2|pp'+qq'|} & \text{in case III-3.} \end{cases}$

The subgroup $\mathcal{D}$ generated by Dehn twists along T is a cyclic normal subgroup of $Sym_0(S^3,L)$ of order $\left|det\begin{bmatrix} p & q \\ q' & p' \end{bmatrix}\right| = |pp'-qq'|$.

(2) If $(p,q) = (p',\pm q')$ and either $L = \tilde{L}$ or $K^{(1)}_{p,q} \cup K^{(2)}_{q',p'}$, then $Sym_0(S^3,L)$ is a subgroup of $Sym(S^3,L)$ of index 2. Otherwise, $Sym(S^3,L) = Sym_0(S^3,L)$.

If $(p,q) \neq (p',-q')$, then the representative $\tilde{L}^*$ of $\tilde{L}$ is given as follows:

$S_1 \cup \psi^{-1}_{p,q}(1/2) \cup \psi^{-1}_{q',p'}(2) \cup S_2$ in Case III-1,4 and 5,

$S_1 \cup \psi^{-1}_{p,q}(1/2) \cup \psi^{-1}_{q',p'}(2) \cup \psi^{-1}_{q',p'}(-2)$ in Case III-2,

$\psi^{-1}_{p,q}(1/2) \cup \psi^{-1}_{p,q}(-1/2) \cup \psi^{-1}_{q',p'}(2) \cup \psi^{-1}_{q',p'}(-2)$ in Case III-3.

If $(p,q) = (p',-q')$, then $\tilde{L}^*$ is obtained from the above by replacing $\psi_{q',p'}$ $(=\psi_{q,-p})$ with $\bar{\psi}_{q,p}$. Here $\bar{\psi}_{q,p}$ is the map $S^3 \longrightarrow \mathbb{C} \cup \{\infty\}$ defined by $\bar{\psi}_{q,p}(z_1,z_2) = (\bar{z}_2)^q / z_1^p$. Note that $\bar{\psi}_{q,p}$ is a quotient map of the S^1 action $\Psi_{q,-p}$. The representative L^* of L is given as a corresponding sublink of $\tilde{L}^*$. To see this, let $\mathcal{D}^*$ be the subgroup of $Isom(S^3,L^*) \cap \mathrm{T}$ consisting of the isometries which preserve each component of L^*. Then $\mathcal{D}^* = Im\Psi_{p,q} \cap Im\Psi_{q',p'} \cong \mathbb{Z}_{|pp'-qq'|}$ and it is generated by $\Psi_{p,q}(\omega)$ (or $\Psi_{q',p'}(\omega)$), where $\omega = exp(2\pi\sqrt{-1}/|pp'-qq'|)$. $\mathcal{D}^*$ gives a realization of $\mathcal{D}$. The subgroup $Isom_0(S^3,L^*) = Isom(S^3,L^*) \cap \langle T,C_1,C_2\rangle$ is equal to

$\langle\Psi_{p,q}(\omega),C_1C_2\rangle$ in Cases Ⅲ-1 and 4,
$\langle\Psi_{p,q}(\sqrt{\omega}),C_1C_2\rangle$ in Case Ⅲ-2,
$\langle\Psi_{p,q}(\sqrt{\omega}),\Psi_{q',p'}(\sqrt{\omega}),C_1C_2\rangle$ in Case Ⅲ-3,
$\langle\Psi_{p,q}(\omega),C_1,C_2\rangle = \langle C_1,C_2\rangle$ in Case Ⅲ-5.

$Isom_0(S^3,L^*)$ realizes $Sym_0(S^3,L^*)$, and if $(p,q) \neq (p',\pm q')$, then it is the full isometry group $Isom(S^3,L^*)$. If $(p,q) = (p',q')$ (resp. $(p,q) = (p',-q')$) and L satisfies the condition (2), then $Isom(S^3,L^*)$ is generated by R (resp. C_1R) and $Isom_0(S^3,L^*)$.

References.

[BZ1] M. Boileau, B. Zimmermann: Symmetries of nonelliptic Montesinos links. Math. Ann. 277 (1987), 563-584.

[BZ2] M. Boileau, B. Zimmermann: The π-orbifold group of a link, to appear in Math. Z.

[BS] F. Bonahon, L. Siebenman: Algebraic knots. Proc. Bangor Conf. 1979, London Math. Soc. Lect. Note Series (to appear).

[CR] P.E. Conner, F. Raymond: Manifolds with few peiodic homeomorphisms. Proc. 2nd Conf. Compact transformation groups, Lect. Notes in Math. 299, Springer-Verlag, 1972, pp.1-75.

[EN] D. Eisenbud, W.D. Neumann: Three-dimensional link theory and invariants of plane curve singularities, Ann. of Math. Studies 110 (1985), Princeton Univ. Press.

[FY] M. Freedman, S.T. Yau: Homotopically trivial symmetries of Haken manifolds are toral. Topology 22 (1983), 179-189.

[GLM] C.McA. Gordon, R.A. Litherland, K. Murasugi: Signatures of covering links, Canad. J. Math. 33 (1981), 381-394.

[GL] C.McA. Gordon, J. Luecke: Knots are determined by their complements, preprint.

[H] M.W. Hirsh: Differential topology, Graduate Text in Math. Springer-Verlag, 1976.

[Ja] W. Jaco: Lectures on three manifold topology, Conference board of Math. Sci., Regional Conference Series in Math. 43, 1980.

[Jo] K. Johannson: Homotopy equivalences of 3-manifolds with boundaries, Lect. Notes in Math. 761, Springer, 1976.

[L] F. Laudenbach: Topologie de la dimension trois homotopie et isotopie, Asterisque 12 (1974).

[Sa] M. Sakuma: Uniqueness of symmetries of knots, Math. Z. 192 (1986), 225-242.

[Si] L. Siebenmann: On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology 3-spheres, Lect. Notes in Math. 788 (1980), 172-222, Springer-Verlag.

[Ta] M. Takahashi: Two-bridge knots have Property P, Memoirs A.M.S. 29 (1981).

[Th1] W.P. Thurston: Three manifolds, Kleinian groups and hyperbolic geometry, Bull. A.M.S. 6 (1982), 357-381.

[Th2] W.P. Thurston: The geometry and the topology of 3-manifolds, to appear in Lect. Note Series, Princeton Univ. Press.

[Th3] W.P. Thurston: Three manifolds with symmetry, preliminary report.

[W] F. Waldhausen: On irreducible 3-manifolds which are sufficiently large, Ann. Math. 87 (1968), 56-88.

[Z1] B. Zimmermann: Das Nielsensche Realizierungsproblem für hinreichend grosse 3-Mannigfaltigkeiten. Math. Z. 180 (1982), 349-359.

[Z2] B. Zimmermann: Finite group actions on Haken 3-manifolds, Quart. J. Math. 37 (1986), 499-511.