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Periods of Composite Links

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Let L be a link in S^3. We call L a periodic link of period n, iff there is an orientation preserving auto-homeomorphism T of S^3 such that

(1) $T(L) = L$,

(2) $Fix(T)$, the fixed point set of T, is a 1-sphere disjoint from L,

(3) T is of period n.

In this paper, we consider the periods of composite links and show that the periods of a composite link are determined by the periods of its prime factors (Theorem 2). In particular, we prove

Theorem. *A nonsplittable composite link can have only a finite number of distinct periods.*

Remark. Trotter [16] and Murasugi [7] proved that a fibered knot can have only a finite number of distinct periods.

As an application, we show that, for each integer $n \geq 2$, there are distinct prime knots whose n-fold branched cyclic coverings are homeomorphic (Theorem 3).

All results given in this paper depend on the following theorem, which is proved using the positive solution of the homotopy Smith Conjecture

[14] and the equivariant sphere theorem of Meeks-Yau [6].

Theorem A. (Theorem 2 (b) of [13]) *Let* L *be a link in* S^3*, and let* M *be a regular covering of* S^3 *branched along* L*. Then* L *is a split link or a composite link, iff* $\pi_2(M) \neq 0$.

1. Periods of composite links

Let L be a periodic link of period n, and let T be the associated periodic map of S^3 of period n. Then, by [14], $Fix(T)$ is a trivial knot and $S^3/T \cong S^3$. Let p be the projection $S^3 \to S^3/T$, $K_0 = p(Fix(T))$, $\underset{\sim}{L} = p(L)$, and $\underset{\sim}{L}' = K_0 \cup \underset{\sim}{L}$. Then we have the following:

Theorem 1.

(a) L *is a split link, iff one of the following conditions holds*:

(1) $\underset{\sim}{L}'$ *is a split link,*

(2) $\underset{\sim}{L}'$ *is a composite link whose decomposing 2-spheres intersect* K_0.

(b) *Assume that* L *is a nonsplittable link. Then,* L *is a composite link, iff* $\underset{\sim}{L}'$ *is a composite link whose decomposing 2-spheres do not intersect* K_0.

(c) L *is a trivial knot, iff* $\underset{\sim}{L}'$ *is the (2,2)-torus link* .

Proof. The "if" parts of (a) and (b) are trivial. We prove the "only if" part of (b). Assume that L is a nonsplittable composite link. Let M be the 2-fold branched cyclic covering of L. Then $\pi_2(M) \neq 0$, by Theorem A. Note that M is the $Z_n \oplus Z_2$ covering of $\underset{\sim}{L}' = K_0 \cup \underset{\sim}{L}$. Then, by Theorem A, we can see that $\underset{\sim}{L}'$ is a split link or a composite link. If $\underset{\sim}{L}'$ is a split link or a composite link whose decomposing 2-spheres intersect K_0, then L is a split link. Hence $\underset{\sim}{L}'$ is a nonsplittable composite link whose decomposing 2-spheres do not intersect K_0, proving the "only if" part of (b). The "only if" part of (a) can be proved by a similar argument as the above, using Theorem 2 (a) of [13]. (c) is proved in [11] and [13].

Theorem 2. *Let* L *be a nonsplittable link, and let* $\#\{n_iL_i \mid 1 \le i \le s\}$ ($s \ge 1$, $n_i \ge 1$) *be the prime decomposition of* L, *where* L_i ($1 \le i \le s$) *is a prime link and* $L_i \ne L_{i'}$ *if* $i \ne i'$ (*see* [3]). *Then, if* n *is a period of* L, *one of the following conditions holds:*

(1) n *devides* n_i *for each* i ($1 \le i \le s$),

(2) *there exists* j ($1 \le j \le s$) *such that*

(i) L_j *is a periodic link of period* n,

(ii) n *divides* $n_j - 1$,

(iii) n *divides* n_i *for each* i ($1 \le i \le s$, $i \ne j$).

Furthermore, if L *is a knot, the above condition is a necessary and sufficient condition for* L *to have period* n.

Proof. Assume that L has period n. Since L is unsplittable, the link $\underset{\sim}{L}'$ is a nonsplittable link and any decomposing 2-sphere does not intersect K_0, by Theorem 1 (a). So $\underset{\sim}{L}'$ is decomposed into prime links $\underset{\sim}{L}'_0 \# \{m_l\underset{\sim}{L}_l \mid 1 \le l \le r\}$ ($r \ge 0$, $m_l \ge 1$), so that $\underset{\sim}{L}'_0 = K_0 \cup \underset{\sim}{L}_0$ and $\underset{\sim}{L} = \underset{\sim}{L}_0 \# \{m_l\underset{\sim}{L}_l \mid 1 \le l \le r\}$ for some link $\underset{\sim}{L}_0$, where $\underset{\sim}{L}_l$ ($1 \le l \le r$) is a prime link and $\underset{\sim}{L}_l \ne \underset{\sim}{L}_{l'}$ if $l \ne l'$ (see Fig. 1 (a)). Then it can be seen that $L = p^{-1}(\underset{\sim}{L}) = p^{-1}(\underset{\sim}{L}_0) \# \{nm_l\underset{\sim}{L}_l \mid 1 \le l \le r\}$ (see Fig. 1 (b)). Since $\underset{\sim}{L}'_0$ is a prime link, $p^{-1}(\underset{\sim}{L}_0)$ is a prime link or a trivial knot by Theorem 1 (b). Thus the above gives the prime decomposition of L, and from this fact, the first half of Theorem 2 follows. The later half follows from the fact that connected sums of knots are unique.

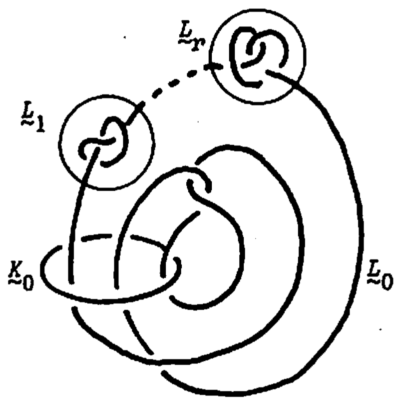

Fig. 1 (a)

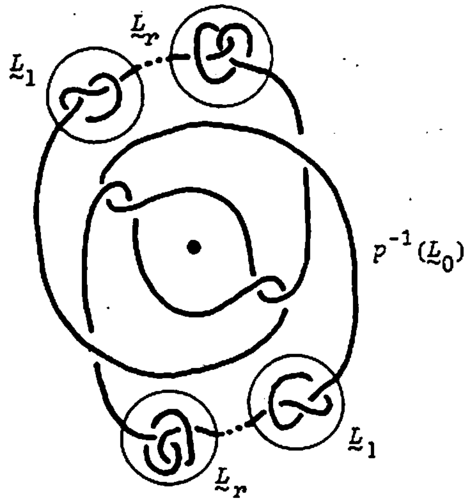

Fig. 1 (b)

Now the theorem in the introduction and the following are immediate.

Corollary. (1) *Let* K *be a (nontrivial) prime knot. Then* $K \# K$ *has period* 2, *and* 2 *is the only period of it.*

(2) *Let* K_1 *and* K_2 *be distinct prime knots. Then* $K_1 \# K_2$ *does not have any periods.*

Example 1. The Granny knot has period 2, and 2 is the only period of it. On the other hand, the square knot does not have any periods.

Example 2. It is known that the only period of the figure eight knot is 2 ([7],[16]). Hence, the periods of the connected sum of n-copies of the figure eight knot are 2 and divisors of n.

2. Application

It is known that there are distinct composite knots all of whose cyclic branched coverings are homeomorphic. Here, we prove the following theorem, which gives an answer to a question of Viro [17].

Theorem 3. *For any integer* $n \geq 2$, *there exist distinct prime knots* J_n *and* K_n *whose n-fold branched cyclic coverings are homeomorphic.*

Remark. For $n = 2$, many examples are known ([2],[15],[17]).

Proof. Consider the 9^2_{35}-link $L = J \cup K$ in the table of [10] (Fig.2).

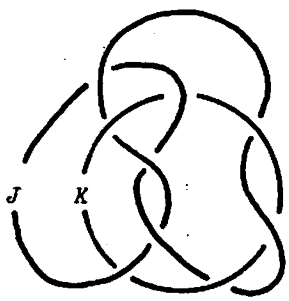

Fig. 2

Note that $lk(J,K) = 1$, both components J and K are unknotted, and therefore, L is prime. Let J_n (resp. K_n) be the inverse image of J (resp. K) in the n-fold branched cyclic covering of K (resp. J). Then, by Theorem 1, J_n and K_n are nontrivial prime knots in S^3. The n-fold branched cyclic coverings of J_n and K_n are homeomorphic to the $Z_n \oplus Z_n$ covering of L; so J_n and K_n have the homeomorphic n-fold branched cyclic coverings.

So, we have only to prove that J_n and K_n are not equivalent.

Let $\Delta_{J_n}(t)$, $\Delta_{K_n}(t)$, and $\Delta_L(x,y)$ be the Alexander polynomials of J_n, K_n, and L respectively. Then, by [7] (cf. [4],[12]),

$$\Delta_{J_n}(t) = \Pi_{i=1}^{n-1}\Delta_L(t,\omega^i) \quad \text{and} \quad \Delta_{K_n}(t) = \Pi_{i=1}^{n-1}\Delta_L(\omega^i,t),$$

where ω is a primitive n-th root of 1, and

$$\begin{aligned}\Delta_L(x,y) = {} & 1 - 2x + 2x^2 - x^3 \\ & -y + 3xy - 3x^2y + 3x^3y - x^4y \\ & -xy^2 + 2x^2y^2 - 2x^3y^2 + x^4y^2.\end{aligned}$$

Since any ω^i $(1 \le i \le n-1)$ is not a root of $1-y$, $deg\Delta_{J_n}(t) = 4(n-1)$. On the other hand, $deg\Delta_{K_n}(t) \le 2(n-1)$. Hence $\Delta_{J_n}(t) \neq \Delta_{K_n}(t)$. This completes the proof of Theorem 3.

Notes. (1) These examples were first exhibited in the master theses of Nakanishi [8] and the auther [11], without proving the primeness of them. Nakanishi [9] gave an elementary proof of the primeness of them by his own method.

(2) In [1], it is announced that Gordon-Litherland had given a similar construction.

(3) Livingston [5] announced that the r-fold branched cyclic coverings of the (p,r)-cable of the (q,r)-torus knot and the (q,r)-cable of the (p,r)-torus knot are homeomorphic. This also proves Theorem 3.

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