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Non-free-periodicity of Amphicheiral Hyperbolic Knots

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A knot K in the 3-sphere S^3 is said to have *free period n* if there is an orientation-preserving homeomorphism f on S^3 such that

(1) $f(K)=K$,

(2) f is a periodic map of period n,

(3) $\mathrm{Fix}(f^i)=\phi$ $(1\leqq i\leqq n-1)$.

Hartley [3] has given very effective methods for determining the free periods of a knot, and has identified the free periods of all prime knots with 10 crossings or less with eight exceptions. Since then, Boileau [1] has calculated the symmetry groups of the "large" Montesinos knots, and has shown that four of the rest have no free periods. The remaining knots are 8_{10}, 8_{20}, 10_{99} and 10_{123} (cf. [5]). By Hartly-Kawauchi [4], 10_{99} and 10_{123} are the only prime knots with 10 crossings or less which are strongly positive amphicheiral. Moreover, it follows from the Theorem of [4] that the polynomial condition given by [3] (Theorem 1.2) does not work for determining whether a strongly positive amphicheiral knot has free period 2 or not.

The purpose of this paper is to prove the following theorem:

Theorem. *Any amphicheiral hyperbolic knot has no free periods.*

In particular, 10_{99} and 10_{123} have no free periods. A circumstantial evidence for this theorem is given by the non-trivial torus knots, which have infinitely many free periods and are not amphicheiral.

§ 1. Some lemmas

Let K be a knot in S^3 which has free period n, and f be a periodic map on S^3 realizing the free period n. Let N be an equivariant tubular neighbourhood of K and put $E=\dot{S}^3-N$.

Lemma 1. *K does not have an f-invariant longitude curve. That is, $f(l)\neq l$, for any simple loop l in ∂N such that $l\sim K$ in $H_1(N)$ and $l\sim 0$ in $H_1(E)$.*

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Proof. See [2] p. 180, where this lemma is proved for the case $n=2$. The same argument works even if $n\geqq 3$.

Lemma 2. *Suppose that K is a hyperbolic knot. Then the restriction of f to $\mathring{E}$ is equivalent to an isometry.*

Proof. Put $E'=E/f$. Then E' is a compact manifold with $\partial E'\cong T^2$, and E' is irreducible since E is so. We show that E' is homotopically atroidal (cf. [6, 14]). Suppose that E' is not homotopically atroidal. Then, by the torus theorem (see [6] p. 156), either E' is a special Seifert fibered space or there is an essential embedding of T^2 in E'. Since E is hyperbolic, E' cannot be a Seifert fibered space. So there is an essential torus T in E'. Then the lift $\tilde{T}$ of T in E is an incompressible torus in E. Since E is hyperbolic, $\tilde{T}$ is boundary parallel, that is, there is a submanifold Q of E, such that $Q\cong T^2\times I$ and $\partial Q=\partial E\cup\tilde{T}$. Q is f-invariant, and Q/f forms a submanifold of E' which is homeomorphic to $T^2\times I$ with $\partial(Q/f)=\partial E'\cup T$; this is a contradiction. Hence E' is homotopically atroidal. Thus, by Thurston [14], $\mathring{E}'$ admits a hyperbolic structure, and therefore, $\mathring{E}$ admits a hyperbolic structure with respect to which f is an isometry.

Lemma 3. *Suppose that (S^3, K) admits an action of $Z_2+Z_2\cong\langle f\,|\,f^2=1\rangle+\langle\gamma\,|\,\gamma^2=1\rangle$, such that*

(1) *f is an orientation-preserving free involution,*

(2) *γ reverses the orientation of S^3.*

Then K is a trivial knot or a composite knot.

Proof. By Livesay [8], S^3/f is homeomorphic to the 3-dimensional projective space P^3. Since f and γ are commutative, γ induces an orientation reversing involution $\underline{\gamma}$ on P^3. Then, by Kwun [7], Fix$(\underline{\gamma})$ is a disjoint union of P^2 and P^0. Let x be a point of $P^2\subset$ Fix(γ) and let $\tilde{x}$ be a lift of x in S^3. Let $\gamma'\colon S^3\to S^3$ be the lift of $\underline{\gamma}$ such that $\gamma'(\tilde{x})=\tilde{x}$. Then Fix$(\gamma')$ contains the inverse image of P^2, which is homeomorphic to a 2-sphere. Thus γ' is a reflection along a 2-sphere. Since γ' is equal to γ or $f\gamma$, γ' preserves the knot K. Hence K must be a trivial knot or a composite knot.

§ 2. Proof of Theorem

Let K be a hyperbolic knot. Then the knot group $G=\pi_1(E)$ is identified with a discrete subgroup of Isom H^3, the isometry group of the 3-dimensional hyperbolic space H^3, and $\mathring{E}$ is identified with H^3/G. We use the upper-half space model $H^3=C\times(0, +\infty)$, and identify Isom H^3

with $P\Gamma L(C)$, the group of all conformal or anti-conformal mappings of the Riemann sphere $C\cup\{\infty\}$, which is identified with the sphere at infinity of H^3. Then the orientation-preserving isometry group $\mathrm{Isom}^+ H^3$ is identified with $PSL(C)$, the group of all Möbius transformations. Let A be the normalizer of G in $P\Gamma L(C)$. Then, by Mostow's rigidity theorem (cf. [14]), the automorphism group $\mathrm{Aut}(G)$ of G is identified with A, and $\mathrm{Isom}\,\mathring{E}\cong\mathrm{Out}(G)$ is identified with A/G. Here, an element $\alpha\in A$ represents the element of $\mathrm{Aut}(G)$ which sends x ($\in G$) to $\alpha x\alpha^{-1}$. Let P be the peripheral subgroup of G generated by a longitude l and a meridian m. Since $P\cong Z+Z$, we may assume that l and m are identified with the Möbius transformations $l(z)=z+\lambda$ and $m(z)=z+1$ respectively, where λ is a complex number with $\mathrm{Im}(\lambda)\neq 0$. Then, as isometries of H^3, we have $l(z, t)=(z+\lambda, t)$ and $m(z, t)=(z+1, t)$, and an end of $\mathring{E}$ is obtained from $C\times[t_0, +\infty)$ by identifying each set $(z+Z\lambda+Z1, t)$ with a point, where t_0 is a sufficiently large number. Let A_∞ be the subgroup of A ($=\mathrm{Aut}(G)$) consisting of those elements which preserve P. Noting that any automorphism of G preserves the subgroups P and $\langle l\rangle$ up to a conjugation, Riley observed the following (see Section 1 of [11]).

Lemma 4. (1) $\mathrm{Isom}\,\mathring{E}\cong A_\infty/P$.

(2) *Any element* ψ *of* A_∞ *is of one of the following types.*

(i) $\psi(z)=z+c$ $(c\in C)$,

(ii) $\psi(z)=-z+c$ $(c\in C)$,

(iii) $\psi(z)=\varepsilon\bar{z}+c$ $(|\varepsilon|=1,\ c\in C)$.

(3) *K is amphicheiral, iff there is an element of A_∞ which is of type* (iii) *with $\varepsilon=\pm 1$, and λ is a purely imaginary number.*

Remark. 5. Let A_∞^* be the subgroup of A_∞ which consists of type (i) elements. Then A_∞^* is a normal subgroup of A_∞; in particular, if $\psi(z)=z+c$ and $\xi(z)=\varepsilon\bar{z}+c'$ $(\varepsilon=\pm 1)$, then $\xi\psi\xi^{-1}(z)=z+\varepsilon\bar{c}$.

Put $\mathrm{Isom}^*\mathring{E}=A_\infty^*/P$. Then, by Smith conjecture [9], we have the following (cf. [10] p. 124, [12] Lemma 3.3).

Lemma 6. $\mathrm{Isom}^*\mathring{E}$ *is a normal subgroup of* $\mathrm{Isom}\,\mathring{E}$ *(of index at most 4), and is isomorphic to a finite cyclic group.*

The proof of the Theorem is divided into two assertions.

Assertion I. *The Theorem is true for free period $n\geqq 3$.*

Proof. Suppose that K is hyperbolic, amphicheiral, and has free period $n\geqq 3$. By Lemma 2, there is an isometry f of $\mathring{E}$ which realizes the free period n. Let ψ be an element of A_∞ representing f (cf. Lemma 4).

Since f preserves a longitude and a meridian homologically, ψ is of type (i); so $\psi(z)=z+c$ for some $c \in C$. Since f has period n, $c=(p\lambda+q1)/n$ for some integers p and q.

Lemma 7. *The greatest common divisors (p, n) and (q, n) are equal to* 1.

Proof. Put $r=n/(p, n)$. Then

$$\begin{aligned}\psi^r(z)&=z+(p\lambda+q1)/(p, n)\\&=l^{p/(p,n)}(z)+q1/(p, n).\end{aligned}$$

Thus the isometry f^r has an invariant meridian curve. (Recall the structure of an end of $\mathring{E}$.) By Smith conjecture [9], we have $f^r=\mathrm{id}$ and therefore $(p, n)=1$. Put $s=n/(q, n)$. Then

$$\begin{aligned}\psi^s(z)&=z+(p\lambda+q1)/(q, n)\\&=m^{q/(q,n)}(z)+p\lambda/(q, n).\end{aligned}$$

Thus the isometry f^s has an invariant longitude curve. So, by Lemma 1, we have $f^s=\mathrm{id}$, and therefore $(q, n)=1$.

Since K is amphicheiral, λ is a purely imaginary number, and $\mathring{E}$ admits an orientation-reversing isometry γ, which is represented by an element ξ of A_∞ such that $\xi(z)=\varepsilon\bar{z}+b$ ($\varepsilon=\pm 1$, $b \in C$) (see Lemma 4). By remark 5,

$$\begin{aligned}\xi\psi\xi^{-1}(z)&=z+\varepsilon(\overline{p\lambda+q1})/n\\&=z+\varepsilon(-p\lambda+q1)/n.\end{aligned}$$

By Lemma 6, there is an integer r ($0\leqq r\leqq n-1$) such that $\gamma f\gamma^{-1}=f^r$, that is, $\xi\psi\xi^{-1}\equiv\psi^r \bmod P$. Hence we have

$$\varepsilon(-p\lambda+q1)/n\equiv r(p\lambda+q1)/n \bmod \{\lambda, 1\}.$$

This is equivalent to

$$\begin{cases}-\varepsilon p\equiv rp \bmod n\\ \ \ \varepsilon q\equiv rq \bmod n.\end{cases}$$

Since $(p, n)=(q, n)=1$ by Lemma 7, we have

$$-\varepsilon\equiv r\equiv\varepsilon \bmod n.$$

This is a contradiction, since $n\geqq 3$. Thus Assertion I is proved.

Assertion II. *The Theorem is true for free period* 2.

Proof. Assume that K is hyperbolic, amphicheiral, and has free period 2. Then Isom* $\mathring{E}$ is a cyclic group of order $2n$ ($n \in N$), and the free period 2 is realized by the isometry $f=f_0^n$, where f_0 is a generator of Isom* $\mathring{E}$. Let ψ_0 be an element of A_∞ representing f_0. Then by an argument similar to the proof of Lemma 7, we can see that $\psi_0(z)=z+(p\lambda+q\,1)/2n$, where p is an integer such that $(p, 2n)=1$ and q is an odd integer. Let ξ be an element of A_∞ representing an orientation-reversing isometry γ of $\mathring{E}$. Then $\xi(z)=\varepsilon\bar{z}+b$ ($\varepsilon=\pm 1$, $b \in C$). Note that $\xi^2(z)=z+(\varepsilon\bar{b}+b)$.

Case 1. $\varepsilon=+1$. Then $\xi^2(z)=z+2\,\mathrm{Re}\,(b)$. Thus γ^2 has an invariant meridian curve, and therefore $\gamma^2=\mathrm{id}$ by Smith conjecture. Since f is the order 2 element of the cyclic normal subgroup Isom* $\mathring{E} \cong Z_{2n}$, we have $\gamma f \gamma^{-1}=f$. So f and γ generate a Z_2+Z_2 action on (S^3, K) which satisfies the condition of Lemma 3. This is a contradiction, since a hyperbolic knot is non-trivial and prime.

Case 2. $\varepsilon=-1$. Then $\gamma^2(z)=z+2\,\mathrm{Im}\,(b)i$. By an argument similar to the final step of the proof of Assertion I, we have $\gamma f_0 \gamma^{-1}=f_0$. Let u be an integer such that $\gamma^2=f_0^u \in$ Isom* $\mathring{E}$.

Subcase 1. u is even. Put $\gamma'=\gamma f_0^{-v}$, where $v=u/2$. Then $(\gamma')^2=\mathrm{id}$. So γ' and f generate a Z_2+Z_2 action on (S^3, K) satisfying the condition of Lemma 3; a contradiction.

Subcase 2. u is odd. Note that

$$\xi^2(z) \equiv \psi_0^u(z)=z+(up\lambda+uq\,1)/2n \mod \{\lambda, 1\}.$$

Since q is odd, $uq/2n \not\equiv 0 \mod 1$, and therefore

$$(up\lambda+uq\,1)/2n \not\equiv a \text{ purely imaginary number} \mod \{\lambda, 1\}.$$

This contradicts the fact that $\xi^2(z)=z+2\,\mathrm{Im}\,(b)i$. This completes the proof of the Theorem.

§ 3. Further discussion

The Theorem does not hold for composite knots. In fact, the connected some of n-copies of an amphicheiral knot is amphicheiral, but has free period n. However, as shown in [13], the free periods of a composite knot are completely determined by the free periods of its prime factors, and the Theorem holds for prime knots except free period 2; that is, any amphicheiral prime knot does not have free periods greater than 2. It remains open whether there is an amphicheiral prime knot which has free period 2.

I also calculated the symmetry groups of the "small" Montesinos knots by using the results of Thurston [15]. In particular, it follows that 8_{10} and 8_{20} have no free periods.* This completes the enumeration of the free periods of the prime knots with 10 crossings or less.

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*) Boileau informed me that he proved the non-free-periodicity of the small Montesinos knots without using [15].

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