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The Homology Groups of Abelian Coverings of Links

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For the infinite cyclic covering space $\tilde{X}_\infty$ of a knot complement, let t be the automorphism of $H_1(\tilde{X}_\infty)$ induced by a generator of the covering transformation group. Then $t-1$ is an isomorphism (Milnor [8]) and the first homology group of the k-fold cyclic branched covering space is isomorphic to $Coker(t^k-1)$ (Gordon [3]).

In this paper we study the universal abelian covering and the cyclic coverings of a link, and establish properties corresponding to the above (Theorems 4 and 6). Furthermore we give a geometrical interpretation to the Hosokawa polynomial (Theorem 1) and simple proofs of the theorems of Hosokawa and Kinoshita [6] about the first homology groups of cyclic branched coverings of a link.

The following notation will be used:

$R[x_1,\cdots,x_n]$: the free R-module with free basis $x_1,\cdots,x_n$,

$< x_1,\cdots,x_n >_R$: the R-submodule generated by $x_1,\cdots,x_n$,

$R[\![x_1,\cdots,x_n]\!]$: the polynomial ring in $x_1,\cdots,x_n$ over R,

$R<x_1,\cdots,x_n>$: the Laurent polynomial ring in $x_1,\cdots,x_n$ over R,

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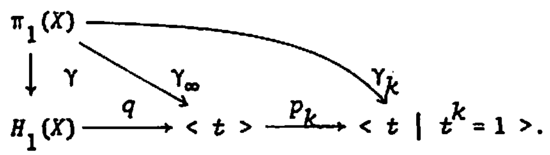
$order_R\, M$: a generator of the order ideal of an R-module M,

$|A|$: the number of the elements of a set A,

$Mat(m,R)$: the ring of (m,m)-matrices over R,

f_*: the homomorphism between homology groups of spaces induced by a continuous map f.

1. Let $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ be an oriented link of μ-components in S^3, N a regular neighbourhood of L, and write $X = S^3 - int\, N$. By Alexander duality the first integral homology group $H_1(X)$ is $< t_1, \cdots, t_\mu \mid t_i t_j = t_j t_i\ 1 \leq i,j \leq \mu >$, where t_i is the meridian of K_i and the linking number $lk(t_i, K_j) = \delta_{i,j}$. We define the group homomorphisms γ, γ_∞, γ_k, q and p_k so that the following diagram is commutative:



Here γ is the abelianization, $q(t_1^{n_1} \cdots t_\mu^{n_\mu}) = t^{n_1 + \cdots + n_\mu}$ and $p_k(t) = t$. The symbol Λ_μ (resp. Λ, Λ_k') denotes the integral group ring of $H_1(X)$ (resp. $< t >$, $< t' \mid t^k = 1 >$), and the ring homomorphism between the group rings induced by a group homomorphism will be written by the same symbol.

Let $\tilde{X}_a$ (resp. $\tilde{X}_\infty$, $\tilde{X}_k$) be the covering space of X corresponding to γ (resp. γ_∞, γ_k) and Σ_k the k-fold cyclic branched covering space of S^3 obtained as the completion of $\tilde{X}_k$. Then by the action of the covering transformation group, $H_*(\tilde{X}_a)$ (resp. $H_*(\tilde{X}_\infty)$, $H_*(\tilde{X}_k)$, $H_*(\Sigma_k)$) has a natural Λ_∞ (resp. Λ, Λ_k', Λ_k') -module structure. The symbol q (resp. p_k) also denotes the natural projection $q : \tilde{X}_a \to \tilde{X}_\infty$ (resp. $p_k : \tilde{X}_\infty \to \tilde{X}_k$).

2. First we consider relations among $H_1(\widetilde{X}_a)$, $H_1(\widetilde{X}_\infty)$ and $H_1(\widetilde{X}_k)$. The fundamental group $\pi_1(X)$ has a presentation:

$$< x_1,\cdots,x_\mu,a_1,\cdots,a_{n-\mu} \mid r_1,\cdots,r_{n-1} >^\phi,$$

such that $\phi(x_i)$ is represented by a meridian of K_i, $\gamma(\phi(x_i)) = t_i$ and $\gamma(\phi(a_j)) = 1$. Let W be the cell complex associated with the presentation. That is, the cell complex W consists of a single vertex e, oriented 1-cells $x_1^*,\cdots,x_\mu^*,a_1^*,\cdots,a_{n-\mu}^*$ and oriented 2-cells $r_1^*,\cdots,r_{n-1}^*$ attached to the 1-skelton according to the relations. Then by the van Kampen theorem there is a canonical isomorphism $\psi : \pi_1(W,e) \to \pi_1(X)$, such that $\psi([x_i^*]) = \phi(x_i^*)$ and $\psi([a_j^*]) = \phi(a_j)$, where $[x_i^*]$ (resp. $[a_j^*]$) is the element of $\pi_1(W,e)$ represented by the oriented loop x_i^* (resp. a_j^*) in W.

Let $\widetilde{W}_a$ (resp. $\widetilde{W}_\infty$, $\widetilde{W}_k$) be the covering space of W corresponding to the group homomorphism $\gamma\circ\psi$ (resp. $\gamma_\infty\circ\psi$, $\gamma_k\circ\psi$). There is also a canonical isomorphism between the first homology groups of $\widetilde{W}_a$ (resp. $\widetilde{W}_\infty$, $\widetilde{W}_k$) and $\widetilde{X}_a$ (resp. $\widetilde{X}_\infty$, $\widetilde{X}_k$). Therefore we identify them from now on.

The chain complex $(C_*(\widetilde{W}_a),\partial_{a,*})$ (resp. $(C_*(\widetilde{W}_\infty),\partial_{\infty,*})$, $(C_*(\widetilde{W}_k),\partial_{k,*})$) associated with the cell complex $\widetilde{W}_a$ (resp. $\widetilde{W}_\infty$, $\widetilde{W}_k$) is a free Λ_μ (resp. Λ, Λ_k') -chain complex. That is, $C_m(\widetilde{W}_a) = 0$ $(m \geq 3)$, $C_2(\widetilde{W}_a) = \Lambda_\mu[\cdots,r_l^*,\cdots]$, $C_1(\widetilde{W}_a) = \Lambda_\mu[\cdots,x_i^*,\cdots,a_j^*,\cdots]$, $C_0(\widetilde{W}_a) = \Lambda_\mu[e]$ and

$$\partial_{a,2}\begin{pmatrix} \vdots \\ r_l^* \\ \vdots \end{pmatrix} = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & \dfrac{\partial r_l}{\partial x_i} & \cdots & \dfrac{\partial r_l}{\partial a_j} & \cdots \\ & \vdots & & \vdots & \end{pmatrix}^{\gamma\circ\phi} \begin{pmatrix} \vdots \\ x_i^* \\ \vdots \\ a_j^* \\ \vdots \end{pmatrix},$$

$$\partial_{a,1}\begin{pmatrix}\vdots\\ x_i^*\\ \vdots\\ a_j^*\\ \vdots\end{pmatrix} = \begin{pmatrix}\vdots\\ x_i - 1\\ \vdots\\ a_j - 1\\ \vdots\end{pmatrix}^{\gamma\circ\phi} (e),$$

where $\partial / \partial x_i$ and $\partial / \partial a_j$ are Fox's free derivatives. Substituting Λ_μ by Λ (resp. Λ_k') and γ by γ_∞ (resp. γ_k), we have the formulas for $(C_*(\widetilde{W}_\infty), \partial_{\infty,*})$ (resp. $(C_*(\widetilde{W}_k), \partial_{k,*})$) (see Gordon [4] Section 6).

Let $q : C_*(\widetilde{W}_a) \to C_*(\widetilde{W}_\infty)$ (resp. $p_k : C_*(\widetilde{W}_\infty) \to C_*(\widetilde{W}_k)$) be the chain map induced by the natural projection $q : \widetilde{W}_a \to \widetilde{W}_\infty$ (resp. $p_k : \widetilde{W}_\infty \to \widetilde{W}_k$).

Lemma 1. *The following hold:*

(i) $Ker\ \partial_{\infty,1} = \Lambda[x_2^* - x_1^*, \cdots, x_\mu^* - x_1^*] \oplus \Lambda[a_1^*, \cdots, a_{n-\mu}^*]$,

(ii) $q(Ker\ \partial_{a,1}) = (t-1)\Lambda[x_2^* - x_1^*, \cdots, x_\mu^* - x_1^*] \oplus \Lambda[a_1^*, \cdots, a_{n-\mu}^*]$,

(iii) $Ker\ \partial_{k,1} = \langle trace_k x_1^* \rangle_{\Lambda_k'} \oplus \Lambda_k'[x_2^* - x_1^*, \cdots, x_\mu^* - x_1^*] \oplus \Lambda_k'[a_1^*, \cdots, a_{n-\mu}^*]$,

where $trace_k = 1 + t^1 + t^2 + \cdots + t^{k-1}$,

(iv) $\langle trace_k x_1^* \rangle_{\Lambda_k'} = \langle trace_k x_1^* \rangle_Z \cong Z$ *where* Z *is the ring of integers,*

(v) $p_k(Ker\ \partial_{\infty,1}) = \Lambda_k'[x_2^* - x_1^*, \cdots, x_\mu^* - x_1^*] \oplus \Lambda_k'[a_1^*, \cdots, a_{n-\mu}^*]$,

(vi) $q(Im\ \partial_{a,2}) = Im\ \partial_{\infty,2}$ *and* $p_k(Im\ \partial_{\infty,2}) = Im\ \partial_{k,2}$.

Proof. (i) By changing basis of $C_1(\widetilde{W}_\infty)$,

$$\partial_{\infty,1}\begin{pmatrix} x_1^*\\ x_2^* - x_1^*\\ \vdots\\ x_\mu^* - x_1^*\\ a_1^*\\ \vdots\\ a_{n-\mu}^*\end{pmatrix} = \begin{pmatrix} t-1\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} (e).$$

(i) follows from the fact that Λ has no divisor of zero.

(ii) Let α be the element of $Ker\ \partial_{a,1}$ written by:

$$\alpha = f_1(t_1,\cdots,t_\mu)x_1^* + \sum_{i=2}^{\mu} f_i(t_1,\cdots,t_\mu)(x_i^* - x_1^*) + \sum_{j=1}^{n-\mu} g_j(t_1,\cdots,t_\mu)a_j^* .$$

Since $\partial_{a,1}(\alpha) = 0$, we have

$$(\#) \quad -f_1(t_1,\cdots,t_\mu)(t_1 - 1) = \sum_{i=2}^{\mu} f_i(t_1,\cdots,t_\mu)(t_i - t_1).$$

By substituting t_i for t in (#), we see $f_1(t,\cdots,t) = 0$. For any fixed j $(2 \le j \le \mu)$, if we replace t_i with t for every i $(i \neq j)$ in (#), we obtain $-f_1(t,\cdots,t_j,\cdots,t)(t - 1) = f_j(t,\cdots,t_j,\cdots,t)(t_j - t)$. Since $Z\langle t,t_j\rangle$ is U.F.D., $t - 1$ divides $f_j(t,\cdots,t_j,\cdots,t)$ and therefore divides $f_j(t,\cdots,t)$. Thus the implication $\subset$ is proved.

To prove the convers implication $\supset$, we have only to show that $(t - 1)(x_i^* - x_1^*)$ belongs to $q(Ker\ \partial_{a,1})$. Let $\beta = (t_1 - 1)x_i^* - (t_i - 1)x_1^*$, then $\partial_{a,1}(\beta) = 0$ and $q(\beta) = (t - 1)(x_i^* - x_1^*)$.

(iii) It will be noticed that $(t - 1)f(t) = 0$ iff $trace_k$ divides $f(t)$, for any $f(t)$ in Λ_k'. Using this fact, (iii) follows from the similar argument to the proof of (i).

(iv) follows from that $t \cdot trace_k = trace_k$ in Λ_k'.

(v) follows from (i).

(vi) follows from the fact that q and p_k are onto.

Theorem 1. *The following hold:*

(i) *The Λ-module $H_1(\tilde{X}_\infty)$ has a square presentation matrix:*

$$A_\infty(t) = \left(\frac{\partial r_l}{\partial x_i}, \frac{\partial r_l}{\partial a_j}\right)^{\gamma_\infty \circ \phi}, \quad 2 \le i \le \mu, \quad 1 \le j \le n - \mu, \quad 1 \le l \le n - 1,$$

and $order_\Lambda H_1(\tilde{X}_\infty) = (t - 1)\Delta(t,\cdots,t)$ *where* $\Delta(t_1,\cdots,t_\mu)$ *is the Alexander polynomial of* L.

(ii) *The* Λ*-module* $q_*H_1(\tilde{X}_a)$ *has a square presentation matrix:*

$$A_a(t) = \left(\frac{1}{t-1}\frac{\partial r_l}{\partial x_i}, \frac{\partial r_l}{\partial a_j}\right)^{\gamma_\infty \circ \phi}, \quad 2 \le i \le \mu, \quad 1 \le j \le n-\mu, \quad 1 \le l \le n-1,$$

and $order_\Lambda H_1(\tilde{X}_a) = \nabla(t)$; *the Hosokawa polynomial of* L (Hosokawa [5]). *Furtheremore the following sequence is exact:*

$$0 \longrightarrow q_*H_1(\tilde{X}_a) \longrightarrow H_1(\tilde{X}_\infty) \longrightarrow (\Lambda / t-1)^{\mu-1} \longrightarrow 0.$$

Proof. (i) follows from Lemma 1 (i) and $det\ A_\infty(t) = (t-1)\Delta(t,\cdots,t)$ (see Murasugi [9] Chapter V Proposition 3.1).

(ii) Since $(t-1)\Lambda[x_2^* - x_1^*, \cdots, x_\mu^* - x_1^*]$ is isomorphic to a free Λ-module $\Lambda[y_2^*, \cdots, y_\mu^*]$ of rank $\mu - 1$ by sending $(t-1)(x_i^* - x_1^*)$ to y_i^*, Lemma 1 (ii) implies the following exact sequence:

$$\Lambda[r_1^*, \cdots, r_{\mu-1}^*] \xrightarrow{A_a(t)} \Lambda[y_2^*, \cdots, y_\mu^*, a_1^*, \cdots, a_{n-\mu}^*] \longrightarrow q_*H_1(\tilde{W}_a) \longrightarrow 0.$$

Hence $A_a(t)$ is a relation matrix for $q_*H_1(\tilde{X}_a)$, and

$$order_\Lambda q_*H_1(\tilde{X}_a) = det\ A_a(t) = (1/t-1)^{\mu-1} det\ A_\infty(t) = (1/t-1)^{\mu-2}\Delta(t,\cdots,t)$$
$$= \nabla(t).$$

From Lemma (i) and (ii), the following sequence is exact:

$$0 \longrightarrow q(Ker\ \partial_{a,1}) \longrightarrow Ker\ \partial_{\infty,1} \longrightarrow (\Lambda / t-1)^{\mu-1} \longrightarrow 0.$$

Factoring this sequence by $q(Im\ \partial_{a,2}) = Im\ \partial_{\infty,2}$, we see that

$$0 \longrightarrow q_*H_1(\tilde{W}_a) \longrightarrow H_1(\tilde{W}_\infty) \longrightarrow (\Lambda / t-1)^{\mu-1} \longrightarrow 0$$

is exact. This completes the proof.

Corollary. *The following hold:*

(i) $rank_\Lambda H_1(\tilde{X}_\infty) = rank_\Lambda q_*H_1(\tilde{X}_a) = nullity\ A_\infty(t) = nullity\ A_a(t)$.

Particularly $H_1(\tilde{X}_\infty)$ *is a torsion* Λ*-module if and only if* $\nabla(t) \neq 0$.

(ii) *If* $\nabla(t) \neq 0$, *then*

(a) $rank_Z H_1(\tilde{X}_\infty) = deg\ \nabla(t) + \mu - 1$,

(b) $H_1(\tilde{X}_\infty)$ *is torsion free as an abelian group, if and only if the greatest common divisor of the coefficients of* $\nabla(t)$ *is equal to* 1.

Proof. These follow immediately from Theorem 1 and the results of Crowell [1].

Theorem 2. (Shinohara and Sumners [12] Theorem 5.2 (i))

$$H_1(\tilde{X}_k) = \langle trace_k x_1^* \rangle_{\Lambda_k'} \oplus p_{k*}H_1(\tilde{X}_\infty) \cong Z \oplus H_1(\tilde{X}_\infty) / (t^k - 1)H_1(\tilde{X}_\infty).$$

Remark. Geometrically, $trace_k x_1^*$ is represented by a meridian of the branch line above K_1.

Proof. The first equality follows from Lemma 1.
The short exact sequence of chain complexes:

$$0 \longrightarrow C_*(\tilde{X}_\infty) \xrightarrow{t^k - 1} C_*(\tilde{X}_\infty) \xrightarrow{p_k} C_*(\tilde{X}_k) \longrightarrow 0$$

implies that $p_{k*}H_1(\tilde{X}_\infty) \cong H_1(\tilde{X}_\infty) / (t^k - 1)H_1(\tilde{X}_\infty)$. This completes the proof.

3. For a knot, $H_2(\tilde{X}_\infty) = 0$ and $t - 1 : H_1(\tilde{X}_\infty) \to H_1(\tilde{X}_\infty)$ is an isomorphism. In this section we study corresponding properties for a link.

We apply the Mayer-Vietoris theorem to $\tilde{C}_\infty$ constructed by using a connected Seifert surface, then obtain the following exact sequence:

$$0 \longrightarrow H_2(\tilde{X}_\infty) \longrightarrow \Lambda^g \xrightarrow{tV - V^T} \Lambda^g \longrightarrow H_1(\tilde{X}_\infty) \longrightarrow 0$$

where V is the Seifert matrix and g is the size of V (see Rolfsen [11]). Since $\underset{Z}{\otimes} Q$ is a exact functor and $Q\langle t\rangle$ is P.I.D., where Q is the field of rational numbers, we have the following:

Theorem 3. *The relation* $H_1(\widetilde{X}_\infty;Q) \cong Q\langle t\rangle^d$ *holds, where* $d = nullity(tV - V^T) = rank_\Lambda H_1(\widetilde{X}_\infty) = nullity\ A_a(t)$.

Theorem 4. *Concerning* $t-1 : H_1(\widetilde{X}_\infty) \to H_1(\widetilde{X}_\infty)$, *the following hold:*

(i) $Ker(t-1) = j_* H_1(\partial\widetilde{X}_\infty)$, *where* j *is the inclusion map* $j : \partial\widetilde{X}_\infty \to \widetilde{X}_\infty$, *and* $rank_Z Ker(t-1) = \mu - 1 - d$,

(ii) $Im(t-1) = q_* H_1(\widetilde{X}_a)$.

Proof. The short exact sequence of chain complexes:

$$0 \longrightarrow C_*(\widetilde{X}_\infty) \xrightarrow{t-1} C_*(\widetilde{X}_\infty) \xrightarrow{p} C_*(X) \longrightarrow 0$$

induces the following long exact sequence:

$$0 \longrightarrow H_2(\widetilde{X}_\infty) \xrightarrow{t-1} H_2(\widetilde{X}_\infty) \xrightarrow{p_{*2}} H_2(X)$$
$$\xrightarrow{\partial_2} H_1(\widetilde{X}_\infty) \xrightarrow{t-1} H_1(\widetilde{X}_\infty) \xrightarrow{p_{*1}} H_1(X) \xrightarrow{\partial_1} H_0(\widetilde{X}_\infty).$$

(i) $H_1(X)$ is a free abelian group of rank $\mu - 1$ generated by $[K_i \times \partial D^2]$ $(1 \le i \le \mu)$, where $K_i \times D^2$ is a regular neighbourhood of K_i and $K_i \times \partial D^2$ is its boundary. Let F be a Seifert surface of L, $K_i' = F \cap (K_i \times \partial D^2)$, and $\widetilde{K}_i'$ a lift of K_i' in $\widetilde{X}_\infty$. From the definition of ∂_2, it can be seen that $\partial_2([K_i \times \partial D^2]) = [\widetilde{K}_i'] \in j_* H_1(\partial\widetilde{X}_\infty)$. Hence $Ker(t-1) = Im\ \partial_2 = j_* H_1(\partial\widetilde{X}_\infty)$.

$$\begin{aligned} rank_Z Ker(t-1) &= rank_Z Im\ \partial_2 = rank_Z H_2(X) - rank_Z Ker\ \partial_2 = \mu - 1 - rank_Z Im\ p_{*2} \\ &= \mu - 1 - rank_Z Coker(t-1 : H_2(\widetilde{X}_\infty) \to H_2(\widetilde{X}_\infty)) \\ &= \mu - 1 - rank_Q Coker(t-1 : H_2(\widetilde{X}_\infty;Q) \to H_2(\widetilde{X}_\infty;Q)). \end{aligned}$$

Since $H_2(\widetilde{X}_\infty;Q) \cong Q\langle t\rangle^d$, $Coker(t-1 : H_2(\widetilde{X}_\infty;Q) \to H_2(\widetilde{X}_\infty;Q)) = Q^d$.

(ii) Let $\alpha = \sum_{i=2}^{\mu} f_i(t)(x_i^* - x_1^*) + \sum_{j=1}^{n-\mu} g_j(t) a_j^*$ be an element of $H_1(\widetilde{X}_\infty)$, then the following hold:

$$\begin{aligned}\alpha \in Im(t-1) &\rightleftarrows \alpha \in Ker\, p_{*1} \\ &\rightleftarrows p_{*1}(\alpha) = \prod_{i=2}^{\mu} (t_i t_1^{-1})^{f_i(1)} = 1 \\ &\rightleftarrows f_i(1) = 0, \quad 2 \le i \le \mu \\ &\rightleftarrows t-1 \text{ divides } f_i(t), \quad 2 \le i \le \mu \\ &\rightleftarrows \alpha \in q_* H_1(\tilde{X}_a).\end{aligned}$$

4. Since Σ_k is the completion of $\tilde{X}_k$,

$$H_1(\Sigma_k) \cong H_1(\tilde{X}_k) / < trace_k x_i^*, \ 1 \le i \le \mu >_Z$$

where $< trace_k x_i^*, \ 1 \le i \le \mu >_Z \cong < trace_k x_i^*, \ 1 \le i \le \mu >_{\Lambda_k'}$.

Theorem 5. (Shinohara and Sumners [12] Theorem 5.4. (i)) *The following sequence is exact:* $0 \longrightarrow Z^{\mu} \longrightarrow H_1(\tilde{X}_k) \longrightarrow H_1(\Sigma_k) \longrightarrow 0.$

Proof. Since the homomorphism $q_{k*} : H_1(\tilde{X}_\infty) \to H_1(X)$, where $q_k : \tilde{X}_\infty \to X$ is the covering projection, maps $trace_k x_i^*$ to t_i^k, $q_{k*} < trace_k x_i^*, \ 1 \le i \le \mu >_Z = < t_k^i, \ 1 \le i \le \mu >_Z \cong Z^{\mu}$. Hence $< trace_k x_i^*, \ 1 \le i \le \mu >_Z \cong Z^{\mu}$. This completes the proof.

Now we obtain the following theorem:

Theorem 6. $H_1(\Sigma_k) \cong H_1(\tilde{X}_\infty) / trace_k H_1(\tilde{X}_\infty)$.

Remark. For a knot, $t-1 : H_1(\tilde{X}_\infty) \to H_1(\tilde{X}_\infty)$ is an isomorphism and $(t-1) trace_k H_1(\tilde{X}_\infty) = (t^k - 1) H_1(\tilde{X}_\infty)$. Hence $H_1(\Sigma_k) \cong H_1(\tilde{X}_\infty) / (t^k - 1) H_1(\tilde{X}_\infty)$ (refer to Gordon [3]).

Proof. From Theorem 2, $H_1(\tilde{X}_k) \cong < trace_k x_1^* >_{\Lambda_k'} \oplus p_{k*} H_1(\tilde{X}_\infty)$ and $< trace_k x_i^*, \ 1 \le i \le \mu >_{\Lambda_k'} = < trace_k x_1^* >_{\Lambda_k'} \oplus < trace_k (x_i^* - x_1^*), \ 2 \le i \le \mu >_{\Lambda_k'}$.

From Lemma 1 (v), $< trace_k(x_i^* - x_1^*),\ 2 \le i \le \mu >_{\Lambda_k'} \subset p_{k*}H_1(\widetilde{X}_\infty)$. Hence

$$\begin{aligned} H_1(\Sigma_k) &\cong H_1(\widetilde{X}_k) / < trace_k x_i^*,\ 1 \le i \le \mu >_{\Lambda_k'} \\ &\cong p_{k*}H_1(\widetilde{X}_\infty) / < trace_k(x_i^* - x_1^*),\ 2 \le i \le \mu >_{\Lambda_k'} \\ &\cong H_1(\widetilde{X}_\infty) / \{< trace_k(x_i^* - x_1^*),\ 2 \le i \le \mu >_\Lambda + (t^k - 1)H_1(\widetilde{X}_\infty)\} . \end{aligned}$$

Thus we have only to prove:

$$< trace_k(x_i^* - x_1^*),\ 2 \le i \le \mu >_\Lambda + (t^k - 1)H_1(\widetilde{X}_\infty) = trace_k H_1(\widetilde{X}_\infty).$$

From Lemma 1 and the fact that $t^k - 1 = (t-1)trace_k$, the implication $\subset$ follows. From Lemma 1 (i), $H_1(\widetilde{X}_\infty)$ is generated by $\{x_i^* - x_1^*,\ a_j^*,\ 2 \le i \le \mu,\ 1 \le j \le n-\mu\}$. From Lemma 1 (ii) and Theorem 4, $a_j^* \in q_*H_1(\widetilde{X}_a) = (t-1)H_1(\widetilde{X}_\infty)$. Hence $a_j^* = (t-1)\alpha$ for some α in $H_1(\widetilde{X}_\infty)$, therefore $trace_k a_j^* = (t^k - 1)\alpha$ is contained in the left hand side. This completes the proof.

5. Now using the previous results, we give alternative proofs of the theorem of Hosokawa [5] on $\nabla(1)$ and the theorems of Hosokawa and Kinoshita [6] on $H_1(\Sigma_k)$. We need the following lemma.

Lemma 2. *Let M be the Λ-module presented by a square matrix $A(t)$ in $Mat(m,\Lambda)$ whose elementary divisors are $e_1(t), e_2(t), \cdots, e_m(t)$ and $det\ A(t) = \Delta(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$ with $a_0a_n \ne 0$. Let $f(t) = c_0t^k + c_1t^{k-1} + \cdots + c_k$ be an element of Λ. Then the following hold:*

(i) *$rank_Z M / f(t)M = \sum_{i=1}^{m} B(e_i(t), f(t))$, where $B(e_i(t), f(t))$ is the number of common roots of $e_i(t)$ and $f(t)$,*

(ii) *If $|c_0| = |c_k| = 1$, then $|M / f(t)M| = |\prod_{l=1}^{k} \Delta(\omega_l)|$ where ω_l $(1 \le l \le k)$ are roots of $f(t)$.*

Proof. (i) Let $\Gamma = \mathbb{C}\langle t\rangle$, where $\mathbb{C}$ is the field of complex numbers. Since Γ is P.I.D., $M \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{i=1}^{m} \Gamma / \langle e_i(t) \rangle_\Gamma$. Therefore $M \otimes_{\mathbb{Z}} \mathbb{C} / f(t)(M \otimes_{\mathbb{Z}} \mathbb{C}) \cong \bigoplus_{i=1}^{m} \Gamma / \langle e_i(t), f(t) \rangle_\Gamma$. If $\omega_{i,j}$ are common roots of $e_i(t)$ and $f(t)$, $\langle e_i(t), f(t) \rangle_\Gamma = \langle \Pi_j (t - \omega_{i,j}) \rangle_\Gamma$. Hence

$$rank_{\mathbb{Z}}\, M / f(t)M = rank_{\mathbb{C}}\, M \otimes_{\mathbb{Z}} \mathbb{C} / f(t)(M \otimes_{\mathbb{Z}} \mathbb{C}) = \sum_{i=1}^{m} B(e_i(t), f(t)).$$

(ii) Let $A(t) = t^d A_0 + t^{d-1} A_1 + \cdots + A_d$, where $A_i \in Mat(m,\mathbb{Z})$. If $|c_0| = |c_k| = 1$, the $\mathbb{Z}$-module $M / f(t)M$ has the following relation matrix R in $Mat(m(k+d),\mathbb{Z})$:

$$R = \left(\begin{array}{cccccccc} A_d & A_{d-1} & \cdots & A_0 & & & & \\ & \ddots & & & \ddots & & & \\ & & & A_d & A_{d-1} & \cdots & & A_0 \\ c_k E & c_{k-1}E & \cdots & c_0 E & & & & \\ & \ddots & & & \ddots & & & \\ & & & c_k E & c_{k-1}E & \cdots & & c_0 E \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}A\\ A\\ A\end{array}}\right\} k\text{-"row"s} \\ \left.\vphantom{\begin{array}{c}A\\ A\\ A\end{array}}\right\} d\text{-"row"s} \end{array}$$

Thus the proof of (ii) is reduced to the following sub-lemma.

Sub-lemma. $det\ R = \pm \prod_{l=1}^{k} \Delta(\omega_l)$.

Proof of the sub-lemma. Consider the polynomial ring $P = \mathbb{Z}[\![\cdots, a_{i,j}^{(s)}, \cdots, \omega_l', \cdots]\!]$ where $1 \le i,j \le m$, $0 \le s \le d$, $1 \le l \le k$, and $P\langle t\rangle$ the Laurent polynomial ring over P in one variable. Let

$A_s' = (a_{i,j}^{(s)}) \in Mat(m,P)$, $0 \le s \le d$,

$A'(t) = t^d A_0' + t^{d-1} A_1' + \cdots + A_d' \in Mat(m, P\langle t\rangle)$,

$\Delta'(t) = det\ A'(t) \in P\langle t\rangle$,

and $f'(t) = \prod_{l=1}^{k} (t - \omega_l') = c_0' t^k + c_1' t^{k-1} + \cdots + c_k' \in P\langle t\rangle$.

Let $R' \in Mat(m(k+d),P)$ be obtained from R by rewriting A_s by A'_s, c_i by c'_i. If we add (the s-th "*column*") $\times$ $(\omega'_l)^{s-1}$ $(2 \le s \le d)$ to the first "*column*", we see:

$$R' \sim \begin{pmatrix} A'(\omega'_l) & \\ \omega'_l A'(\omega'_l) & \\ \vdots & \\ \omega_l'^{k-1} A'(\omega'_l) & * \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix} \sim \begin{pmatrix} A'(\omega'_l) & \\ 0 & \\ \vdots & *' \\ \vdots & \\ 0 & \end{pmatrix}.$$

Hence $\Delta'(\omega'_l) = det\ A'(\omega'_l)$ divides $det\ R'$ in P for each l $(1 \le l \le k)$. Since $\Delta'(\omega'_l)$ $(1 \le l \le k)$ are relatively prime, $\prod_{l=1}^{k} \Delta'(\omega'_l)$ divides $det\ R'$ in P. Comparing the coefficients, we see $det\ R' = \prod_{l=1}^{k} \Delta'(\omega'_l)$. The proof of Lemma 2 is complete.

Let us study $\nabla(1)$. From Theorem 1 and Lemma 2,

$$|\nabla(1)| = |q_* H_1(\tilde{X}_a) \,/\, (t-1) q_* H_1(\tilde{X}_a)|.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & j_* H_1(\partial\tilde{X}_\infty) & \longrightarrow & H_1(\tilde{X}_\infty) & \xrightarrow{t-1} & q_* H_1(\tilde{X}_a) & \longrightarrow & 0 \\
& \downarrow{\scriptstyle t-1} & & \downarrow{\scriptstyle t-1} & & \downarrow{\scriptstyle t-1} & & \\
0 \longrightarrow & j_* H_1(\partial\tilde{X}_\infty) & \longrightarrow & H_1(\tilde{X}_\infty) & \xrightarrow{t-1} & q_* H_1(\tilde{X}_a) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& j_* H_1(\partial\tilde{X}_\infty) & \longrightarrow & H_1(\tilde{X}_\infty) / (t-1) H_1(\tilde{X}_\infty) & \longrightarrow & q_* H_1(\tilde{X}_a) / (t-1) q_* H_1(\tilde{X}_a) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& 0 & & 0 & & 0 & &
\end{array}$$

From Theorem 4 the first and the second rows are exact. Since $t-1 = 0 : H_1(\partial\tilde{X}_\infty) \to H_1(\partial\tilde{X}_\infty)$, the first column is exact, and obviously the other columns are also exact. Hence the third row is exact (see, for example, MacLane [7] p.50). Since $H_1(\widetilde{X}_\infty) / (t-1)H_1(\widetilde{X}_\infty) \cong Ker(\partial_1 : H_1(X) \to H_0(\widetilde{X}_\infty))$ from the exact sequence in the proof of Theorem 4, we see:

$$q_*H_1(\widetilde{X}_a) / (t-1)q_*H_1(\widetilde{X}_a) \cong Ker\ \partial_1 / Im(p_*\circ j_*),$$

where

$$p_*\circ j_* \begin{pmatrix} [\widetilde{K}'_1] \\ \vdots \\ [\widetilde{K}'_\mu] \end{pmatrix} = U \begin{pmatrix} t_1 \\ \vdots \\ t_\mu \end{pmatrix}, \quad \partial_1 \begin{pmatrix} t_1 \\ \vdots \\ t_\mu \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} (e),$$

and $U = (\, l_{i,j}\,),\quad l_{i,j} = \begin{cases} lk(K_i, K_j) & (i \neq j) \\ -\sum_{s\neq i} lk(K_i, K_s) & (i = j) \end{cases}.$

From the above it follows that any principal sub-matrix of U is a relation matrix of $q_*H_1(\widetilde{X}_a) / (t-1)q_*H_1(\widetilde{X}_a)$. Thus we have the following:

Theorem 7. (Hosokawa [5] Theorem 1) $\pm\nabla(1)$ *is equal to any principal minor determinant of* U.

Next we study $H_1(\Sigma_k)$. Consider the following exact sequence:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & q_*H_1(\widetilde{X}_a) & \longrightarrow & H_1(\widetilde{X}_\infty) & \longrightarrow & (\Lambda / t-1)^{\mu-1} & \longrightarrow & 0 \\
 & & \downarrow trace_k & & \downarrow trace_k & & \downarrow trace_k & & \\
0 & \longrightarrow & q_*H_1(\widetilde{X}_a) & \longrightarrow & H_1(\widetilde{X}_\infty) & \longrightarrow & (\Lambda / t-1)^{\mu-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & q_*H_1(\widetilde{X}_a) / trace_k q_*H_1(\widetilde{X}_a) & \longrightarrow & H_1(\Sigma_k) & \longrightarrow & (Z / kZ)^{\mu-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array}$$

The first and the second rows are exact from Theorem 1, the second column is exact from Theorem 6, and obviously the first and the third columns are exact. Therefore the third row is exact. From Theorem 1 and Lemma 2, we obtain the following theorem.

Theorem 8. (Hosokawa and Kinoshita [6] Theorems 1 and 2)

(i) $|H_1(\Sigma_k)| = k^{\mu-1}|\prod_{j=1}^{k-1}(\omega_j)|$, *where* ω_j *are k-th roots of* 1 *distinct from* 1.

(ii) $rank_Z\, H_1(\Sigma_k) = \sum_{i=1}^{n-1} B(e_{a,i}(t), trace_k)$, *where* $e_{a,i}(t)$ *are elementary divisors of* $A_a(t)$.

Remark. From Theorems 2 and 5 and Lemma 2, we see that $rank_Z\, H_1(\Sigma_k) = \sum_{i=1}^{n-1} B(e_{\infty,i}(t), t^k - 1) - (\mu - 1)$, where $e_{\infty,i}(t)$ are elementary divisors of $A_\infty(t)$ (Theorem 2 in [6]).

Corollary. $rank_Z\, H_1(\Sigma_k) \geq B(\nabla(t), trace_k)$.

Remark. Though Theorem 3 in [6] says that $rank_Z H_1(\Sigma_k) \geq B(\nabla(t), t^k - 1)$, it is incorrect. For example, Whitehead link has $A_a(t) = ((t-1)^2)$ and $rank_Z\, H_1(\Sigma_k) = B((t-1)^2, trace_k) = 0$, but $B((t-1)^2, t^k - 1) = 1$.

We close this paper by proving the following result obtained by Murasugi and Mayberry [10].

Theorem 9. *If* $rank_Z H_1(\Sigma_k) = 0$, *then* $|Tor_Z\, H_1(\tilde{X}_k)| = \left|\prod_{j=1}^{k-1} \nabla(\omega_j)\right|$.

Proof. From Theorem 2, $Tor_Z H_1(\tilde{X}_k) \cong Tor_Z(H_1(\tilde{X}_\infty) / (t^k - 1)H_1(\tilde{X}_\infty))$. By the similar argument to the proof of Theorem 8, we obtain the following exact sequence:

$$q_* H_1(\tilde{X}_a) / (t^k - 1)q_* H_1(\tilde{X}_a) \longrightarrow H_1(\tilde{X}_\infty) / (t^k - 1)H_1(\tilde{X}_\infty) \longrightarrow Z^{\mu-1} \longrightarrow 0.$$

Since $(t^k - 1)H_1(\tilde{X}_\infty) = trace_k q_* H_1(\tilde{X}_a)$ from Theorem 4, the following sequence is exact:

$$0 \longrightarrow q_*H_1(\tilde{X}_a) \,/\, trace_k q_*H_1(\tilde{X}_a) \longrightarrow H_1(\tilde{X}_\infty) \,/\, (t^k - 1)H_1(\tilde{X}_\infty) \longrightarrow Z^{\mu-1} \longrightarrow 0.$$

On the other hand, from the proof of Theorem 8, we see that $q_*H_1(\tilde{X}_a) / trace_k q_*H_1(\tilde{X}_a)$ is finite iff $rank_Z H_1(\Sigma_k) = 0$. Hence $Tor_Z H_1(\tilde{X}_k) \cong q_*H_1(\tilde{X}_a) / trace_k q_*H_1(\tilde{X}_a)$, if $rank_Z H_1(\Sigma_k) = 0$. Now this theorem follows from Lemma 2.

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