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DEHN SURGERY ON SYMMETRIC KNOTS

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For a knot K in S^3 and a rational number r, let $M(K, r)$ be the closed 3-manifold obtained by Dehn surgery of type r on K. K is said to have *Property* $P_n(n \in Z\backslash\{0\})$, iff $M(K, 1/n)$ is not simply connected, and K is said to have *Property P*, iff K has Property P_n for any $n \in Z\backslash\{0\}$. A link L in S^3 is said to have *Property P**, iff every Dehn surgery of S^3 on L does not produce a fake homotopy 3-sphere (cf. [23]).

Montesinos [13] studied the relationship between 2-fold branched coverings and closed 3-manifolds obtained by Dehn surgery on links with symmetry. In particular, he showed that every simply connected 2-fold branched covering of S^3 is homeomorphic to S^3, iff every strongly invertible link has Property P*. (See p. 227 of [13]. Note that the term "Property P" in [13] means Property P* in this paper.) Hence, by the positive solution of the homotopy Smith conjecture [20], it follows that every strongly invertible link has Property P*. Nevertheless, it remains open whether every strongly invertible knot has Property P.

In this paper, using the techniques of Montesinos [13] and the homotopy Smith conjecture, we prove the Property P conjecture for some classes of knots with symmetry — a class containing all 3-strand pretzel knots of odd type (Theorem 1), and a class containing all 2-bridge knots (Theorem 2). (For 2-bridge knots, the Property P conjecture has been proved by Takahashi [21].)

We apply our method to the knots in the knot table which were not proved by Riley [17] to have Property PP (which is stronger than Property P). Through it, we can conclude that all knots in the table with 9 crossings or less except 8_{17}, 9_{32} and 9_{33} have Property P (see Section 7). (The exceptional knots 8_{17}, 9_{32} and 9_{33} are the only knots with 9 crossings or less, which are non-invertible (see [5, 7]).)

1. Dehn surgery on periodic knots

Let $L=O \cup K$ be a 2-component link in S^3 with O a trivial knot, and let $n(n>1)$ be a positive integer relatively prime to $\lambda=lk(O, K)$. The n-fold cyclic branched covering $\sum_n(O)$ of S^3 branched along O is again a 3-sphere, and the lift, $C_n(L)$, of K to $\sum_n(O)$ is a periodic knot of period n. (By [20], every periodic knot is so obtained.) Let $N(K)$ be a regular neighbourhood of K, which is disjoint from O. For an integer k, the homology sphere $M(K, 1/k)$ is obtained from $S^3 - int\, N(K)$ by sewing a solid torus. Let $W_k(L)$ be the knot in $M(K, 1/k)$ given by $O \subset S^3 - int\, N(K) \subset M(K, 1/k)$. [Hereafter, for a 2-component ordered link $L = K_1 \cup K_2$ in S^3, we use the symbol $C_n(L)$ (resp. $W_k(L)$) to denote the lift of K_2 in $\Sigma_n(K_1)$ (resp. the knot K_1 in $M(K_2, 1/k)$).]

The following is a generalization of Theorem 2 of [13].

PROPOSITION 1. *For an integer q, $M(C_n(L), 1/q)$ is the n-fold cyclic branched covering of $M(K, 1/nq)$ branched along $W_{nq}(L)$.*

PROOF. Let p be the covering projection $\Sigma_n(O) \to S^3$, and let $\tilde{N}(K) = p^{-1}(N(K))$. Then $\tilde{N}(K)$ is a regular neighbourhood of $C_n(L)$, which is invariant under the Z_n-action on $\Sigma_n(O)$. Let ℓ and m be a preferred longitude (see p. 31 of [19]) and a meridian of $N(K)$, and let $\tilde{\ell}$ and $\tilde{m}$ be a lift of ℓ and m respectively. Then $\tilde{\ell}$ and $\tilde{m}$ are a preferred longitude and a meridian of $\tilde{N}(K)$ respectively, and their homology classes satisfy the equations $p_*([\tilde{\ell}]) = n[\ell]$ and $p_*([\tilde{m}]) = [m]$ in $H_1(\partial N(K))$. Now, $M(C_n(L), 1/q)$ is obtained from $\Sigma_n(O) - int\, \tilde{N}(K)$ and a solid torus $\tilde{T}$ by identifying their boundaries, where a meridian $\tilde{\mu}$ of $\tilde{T}$ is identified with a simple loop on $\partial\tilde{N}(K)$ representing the homology class $q[\tilde{\ell}] + [\tilde{m}] \in H_1(\partial\tilde{N}(K))$. It can be seen that the Z_n-action on $\partial\tilde{N}(K) = \partial\tilde{T}$ extends to a free Z_n-action on $\tilde{T}$, such that $\tilde{T}/Z_n$ is again a solid torus T. Thus we obtain a Z_n-action on $M(C_n(L), 1/q)$, such that

(1) $Fix(Z_n) = p^{-1}(O)$, and

(2) $$M(C_n(L), 1/q)/Z_n = (\Sigma_n(O) - int\, \tilde{N}(K))/Z_n \cup \tilde{T}/Z_n \cong (S^3 - int\, N(K)) \cup T.$$

Here a meridian μ of T is identified with a simple loop on $\partial N(K)$ representing the homology class $nq[\ell] + [m] \in H_1(\partial N(K))$, since $p_*([\tilde{\mu}]) = p_*(q[\tilde{\ell}] + [\tilde{m}]) = nq[\ell] + [m]$ in $H_1(\partial N(K))$. This completes the proof.

From the above, we have the following proposition.

PROPOSITION 2. *$C_n(L)$ has Property P_q ($q \in Z \backslash \{0\}$), iff K has Property P_{nq} or the knot $W_{nq}(L)$ in $M(K, 1/nq)$ is non-trivial.*

PROOF. This follows from the positive solution of the homotopy Smith Conjecture [20] and the fact that the homomorphism $\pi_1(M(C_n(L), 1/q)) \to \pi_1(M(K, 1/nq))$ induced by the covering projection is an epimorphism.

EXAMPLE 1. The knot 8_{18} is a periodic knot of period 4. Let h be a periocid map on S^3 realizing the 4-fold symmetry. Then the knot $8_{18}/h^2 \subset S^3/h^2 \cong S^3$ is a figure-eight knot, which has Property P by [1, 3]. Hence 8_{18} has Property P.

EXAMPLE 2. Let $L=O \cup K$ be a link as illustrated in Fig. 1. Then $C_2(L)$ is a figure-eight knot (see Fig. 2).

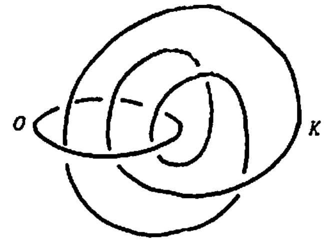


Fig. 1

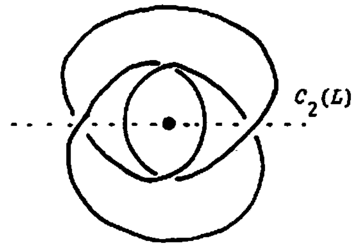


Fig. 2

Since K is a trivial knot, $M(C_2(L), 1/q)$ is the 2-fold branched covering of S^3 branched along the knot $W_{2q}(L)$, which is obtained from O by $(-2q)$ right-hand full twists along K (see Fig. 3).

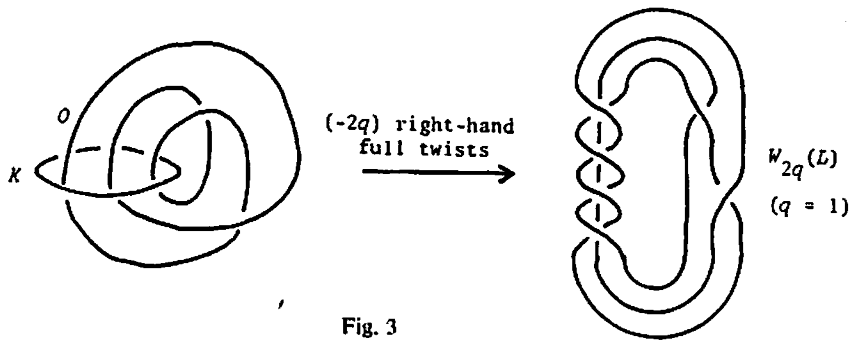


Fig. 3

The Alexander polynomial of $W_{2q}(L)$, which is calculated in Example 4 of Section 3, is nontrivial. So, we can conclude that the figure-eight knot has Property P.

2. Dehn surgery on strongly invertible knots

Let O be a trivial knot in S^3, and let J be an arc in S^3 such that $J \cap O = \partial J$. The 2-fold branched covering $\Sigma_2(O)$ of S^3 branched along O is a 3-sphere, and the inverse image, $I(J)$, of the arc J in $\Sigma_2(O)$ is a strongly invertible knot. (By [24], every strongly invertible knot is so obtained.)

Montesinos proved that $M(I(J), r)$ $(r \in Q \cup \{\infty\})$ is a 2-fold branched covering of S^3 (see Theorem 1 of [13]). Let $F_q(J)$ $(q \in Z)$ be the branch line of the branched covering $M(I(J), 1/q) \to S^3$. Then, by the homotopy Smith conjecture [20], we have the following.

PROPOSITION 3. *$I(J)$ has Property P_q ($q \in Z \setminus \{0\}$), iff the knot $F_q(J)$ is nontrivial.*

Here, we describe the branch line $F_q(J)$ according to [14] (cf. [12]). Let B be a regular neighbourhood of J, such that O intersects B in two disjoint proper arcs (see Fig. 4). Then the inverse image $\tilde{B}$ of B is a regular neighbourhood of $I(J)$, which is invariant under the nontrivial covering transformation t (see Fig. 5).

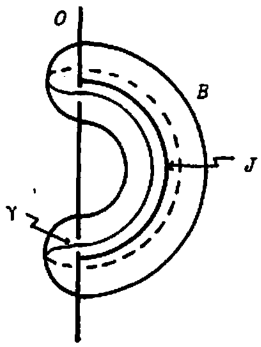


Fig. 4

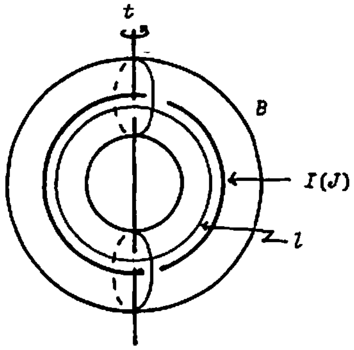


Fig. 5

Let ℓ be a preferred longitude of $\tilde{B}$, such that $t(\ell) \cap \ell = \emptyset$, and let $\gamma = p(\ell)$, where p is the projection $\sum_2(O) \to S^3$ (see Fig.'s 4, 5 and 6). Let D_γ be a homeomorphism on B as illustrated in Fig. 6.

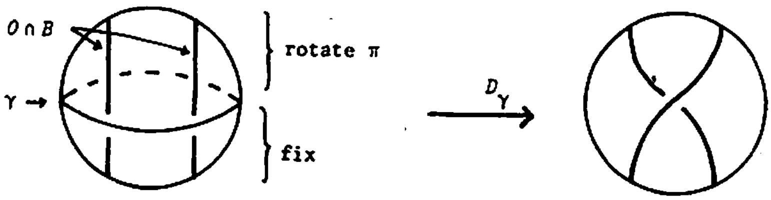


Fig. 6

Then the branch line $F_q(J)$ is given by

$$F_q(J) = (O - (O \cap int\, B)) \cup D_\gamma^{-q}(O \cap B) \subset (S^3 - B) \cup B = S^3.$$

Let $L_0(J) = O \cup \gamma$ and $L_1(J) = F_1(J) \cup \gamma$. Then, we have

$$F_q(J) \cong \begin{cases} W_k(L_0(J)) & (q = 2k) \\ W_k(L_1(J)) & (q = 2k+1). \end{cases}$$

(Recall the definition of $W_k(\ )$ given in Section 1.)

EXAMPLE 3. Let O and J be a trivial knot and an arc in S^3 as illustrated in Fig. 7. Then $I(J)$ is a figure-eight knot (see Fig. 8).

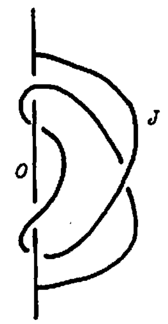


Fig. 7

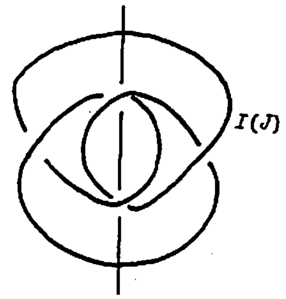


Fig. 8

The knot $F_q(J)=(O-(O\cap int\, B))\cup D_\gamma^{-q}(O\cap B)$ is illustrated in Fig. 9.

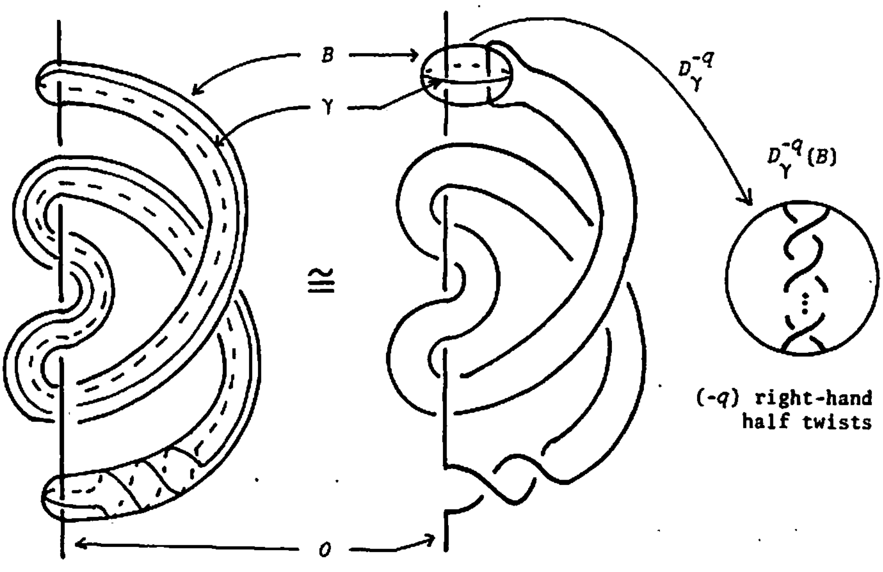


Fig. 9

The Alexander polynomial of $F_q(J)$, which is calculated in Example 5 of Section 3, is nontrivial. So, we can again conclude that the figure-eight knot has Property P.

3. The effect of the transformation $W_k(\quad)$ on the Alexander polynomials

As discussed in the previous sections, the Property P conjecture for periodic knots and strongly invertible knots is reduced to proving the nontriviality of knots obtained from certain 2-component links through the operation $W_k(\ )$.

In this section, we give formulas of the Alexander polynomials of such knots, one of which was formulated by Kidwell [8].

Let $L=K_1 \cup K_2$ be an oriented ordered link in an oriented S^3. Recall that $W_k(L)$ denotes the knot K_1 in $M(K_2, 1/k)$. If K_2 is a trivial knot, $W_k(L)$ is a knot in S^3 obtained from K_1 by $(-k)$ right-hand full twists along K_2. Let $\Delta(x, y)$ be the Alexander polynomial of the link L, $\lambda=lk(K_1, K_2)$, and $\Delta_k(t)$ be the Alexander polynomial of the knot $W_k(L)$. In case $\lambda=0$, define a polynomial $A(t)$ as follows. Let $V=M(K_2, 0)-int\ N(K_1)$, where $N(K_1)$ is a regular neighbourhood of K_1, and let $\tilde{V}$ be the infinite cyclic cover of V corresponding to the composite homomorphism $\pi_1(V)\to Z$ of the abelianization and τ: $H_1(V)\cong Z\oplus Z\to Z$ where τ carries the meridians of K_1 and K_2 to a generator and zero respectively. Define $A(t)$ to be the determinant of a square presentation matrix of the $Z\langle t\rangle$-module $H_1(\tilde{V})$ (see Section 2 of [9]). We call $A(t)$ the $A\tau$-polynomial of L.

PROPOSITION 4. (1) (Corollary 3.2 of Kidwell [8]) *In case $\lambda\neq 0$, we have $\Delta_k(t)=\Delta(t, t^{-k\lambda})/\rho_\lambda(t)$, where $\rho_\lambda(t)=(t^\lambda-1)/(t-1)$.*

(2) *In case $\lambda=0$, we have $\Delta_k(t)=\Delta_0(t)+kA(t)$.*

REMARK. Since the polynomials $\Delta_k(t)$ and $A(t)$ are well-defined only up to units of the group ring $Z\langle t\rangle$, there remains some ambiguity in the formula (2). The precise meaning of it is as follows: For a suitable representation of the polynomials

$$A(t) = a_0 + \sum_{i>0} a_i(t^i+t^{-i}) \quad \text{and} \quad \Delta_0(t) = b_0 + \sum_{i>0} b_i(t^i+t^{-i}),$$

we have $\Delta_k(t)=(b_0+ka_0)+\sum_{i>0}(b_i+ka_i)(t^i+t^{-i})$.

PROOF. (1) This is a generalization of the Torres' formula, and, in fact, is proved by Kidwell [8] by using it.

(2) . To prove this formula, we use arguments of Kojima-Yamasaki [9] and Rolfsen [18]. By [18], there are disjoint solid tori $T_1,\dots, T_n$ in S^3 and a self-homeomorphism h on $S^3-int(T_1\cup\cdots\cup T_n)$, such that

(1) $h(K_1)$ is unknotted in S^3,
(2) $lk(T_r, K_1)=lk(T_r, h(K_1))=0$ for all r,
(3) $h(\partial T_r)=\partial T_r$ and $lk(\mu_r', T_r)=\pm 1$, where μ_r is a meridian of T_r and $\mu_r'=h(\mu_r)$.

Since $h(K_1)$ is unknotted, the infinite cyclic cover of $S^3-int\ h(N(K_1))$ is

$$p: R^1\times D^2 \longrightarrow S^1\times D^2 \cong S^3 - int\ h(N(K_1)).$$

Since $lk(h(K_1), K_2)=0$ (resp. $lk(T_r, h(K_1))=0$), a lift $\tilde{N}(K_2)$ (resp. $\tilde{T}_r$) of $N(K_2)$, a regular neighbourhood of K_2 (resp. T_r), is homeomorphic to $D^2\times S^1$. Let t be a generator of the covering transformation group, ℓ and m be a preferred longitude and a meridian of K_2 respectively, and $\tilde{\ell}$ (resp. $\tilde{m}$, $\tilde{\mu}'_r$) be the lift of ℓ (resp. m, μ'_r) to $\partial\tilde{N}(K_2)$ (resp. $\partial\tilde{N}(K_2)$, $\partial\tilde{T}_r$). Let $E_k=M(K_2, 1/k)-int\ N(W_k(L))$, where $N(W_k(L))$ is a regular neighbourhood of the knot $W_k(L)$, and let $\tilde{E}_k$ be the infinite cyclic cover of E_k. Then $\tilde{V}$ (resp. $\tilde{E}_k$) is obtained from $R^1\times D^2$ by removing each $int\ t^i(\tilde{N}(K_2))$, $int\ t^j(\tilde{T}_r)$ $(i, j\in Z)$, and sewing back a solid torus so that its meridian coincides with $t^i\tilde{\ell}$ (resp. $t^i(k\tilde{\ell}+\tilde{m})$) or $t^j\tilde{\mu}'_r$ (resp. $t^j\tilde{\mu}'_r$). Then, by Proposition 4 of [9], $H_1(\tilde{V})$ has a presentation matrix

$$\begin{pmatrix} a & b \\ c^T & D \end{pmatrix}$$

where $b=(b_1,\dots, b_n)$, $c=(c_1,\dots, c_n)$, $D=(d_{sr})_{1\leq s,r\leq n}$

with $a=\sum_i lk(\tilde{\ell}, t^i\tilde{K}_2)t^i$, $b_r=\sum_i lk(\tilde{\ell}, t^i\tilde{T}_r)t^i$, $c_s=\sum_i lk(\tilde{\mu}'_s, t^i\tilde{K}_2)t^i$, $d_{sr}=\sum_i lk(\tilde{\mu}'_s, t^i\tilde{T}_r)t^i$.

Here $\tilde{K}_2$ is the lift of K_2 to $\tilde{N}(K_2)$, and $lk(\ ,\ )$ is the linking number in $R^1\times D^2$. By [18], D is a presentation matrix of $H_1(\tilde{E}_0)$. Recall that, in constructing $\tilde{E}_k$, the meridian of the solid torus attached to $\partial\tilde{N}(K_2)$ is identified with $k\tilde{\ell}+\tilde{m}$. From this fact, we can see that $H_1(\tilde{E}_k)$ has a presentation matrix

$$\begin{pmatrix} 1+ka & kb \\ c^T & D \end{pmatrix}.$$

Hence
$$\begin{aligned} \Delta_k(t) &= det\begin{pmatrix} 1+ka & kb \\ c^T & D \end{pmatrix} \\ &= det\begin{pmatrix} 1 & kb \\ 0 & D \end{pmatrix} + det\begin{pmatrix} ka & kb \\ c^T & D \end{pmatrix} \\ &= det\ D + k\ det\begin{pmatrix} a & b \\ c^T & D \end{pmatrix} \\ &= \Delta_0(t) + kA(t). \end{aligned}$$

This completes the proof. (The statement in the remark follows from the fact that $\Delta_k(t)$ and $A(t)$ are symmetric (see [6]).)

EXAMPLE 4. Consider the same setting as that of Example 2. Then $\lambda= lk(O, K)=3$, and the Alexander polynomial $\Delta(x, y)$ of $L=O\cup K$ is $x+(1-x+$

$x^2)y+xy^2$. Hence, by Proposition 4, the Alexander polynomial $\Delta^{(q)}(t)$ of the knot $W_{2q}(L)$ is given by

$$\Delta^{(q)}(t) = \Delta(t, t^{-6q})/(1+t+t^2) = (t+t^{-6q}-t^{-6q+1}+t^{-6q+2}+t^{-12q+1})/(1+t+t^2).$$

In particular, $deg\, \Delta^{(q)}(t) = 12|q|-2$ $(q \neq 0)$, and therefore $\Delta^{(q)}(t) \neq 1$.

EXAMPLE 5. Consider the same setting as that of Example 3. Recall that the knot $F_q(J)$ is equivalent to $W_k(L_0(J))$ or $W_k(L_1(J))$ according to whether $q=2k$ or $q=2k+1$, where $L_0(J)=O \cup \gamma$ and $L_1(J)=F_1(J) \cup \gamma$. Here $lk(O, \gamma) = lk(F_1(J), \gamma)=0$. Hence, by Proposition 3, the Alexander polynomial $\Delta^{(q)}(t)$ of the knot $F_q(J)$ is given as follows.

$$\Delta^{(q)}(t) = \begin{cases} 1+kA_0(t) & (q=2k) \\ \Delta^{(1)}(t) + kA_1(t) & (q=2k+1), \end{cases}$$

where $A_0(t)$ and $A_1(t)$ are the $A\tau$-polynomials of the links $L_0(J)$ and $L_1(J)$ respectively. By direct calculation, we have

$$\begin{aligned} A_0(t) &= [-2, 1, 1, -1 \\ \Delta^{(1)}(t) &= [1, 0, -1, 1, 0, -1, 1 \\ A_1(t) &= [0, 1, -2, 1, 1, -2, 1, \end{aligned}$$

where $[a_0, a_1, \dots, a_n$ means $a_0 + \sum_{i=1}^{n} a_i(t^i + t^{-i})$. In particular, $deg\, \Delta^{(q)}(t)$ $(q \neq 0)$ is equal to 6, 10 or 12, according to whether q is even, -1 or one of the rest, and therefore $\Delta^{(q)}(t) \neq 1$.

REMARK. Since the figure-eight knot is amphicheiral, we have $M_q \cong M_{-q}$, where $M_q = M$ (figure-eight, $1/q$). So the knots $W_{\pm 2q}(L)$ in Example 2 and the knots $F_{\pm q}(J)$ in Example 3 have the same homology 3-sphere M_q as 2-fold branched coverings. On the other hand, by Examples 4 and 5, we have

(1) $W_{2q}(L) \cong W_{-2q}(L)$ for any q,

(2) $F_q(J) \not\cong F_{-q}(J)$ for any q,

(3) $F_q(J) \not\cong W_{\pm 2q}(L) \not\cong F_{-q}(J)$, if $|q| > 1$,

(4) $F_1(J) \not\cong W_{\pm 2}(L) \cong F_{-1}(J)$.

Hence, if $|q|>1$ (resp. $|q|=1$), there are three (resp. two) inequivalent knots in S^3 whose 2-fold branched coverings are homeomorphic to the same homology 3-sphere M_q. Takahashi [22] constructed such knots from a different point of view. In fact, it can be seen that they are $F_q(J)$ and $F_{-q}(J)$.

At the end of this section, we explain a convenient method for calculating Alexander polynomials of 2-component links given by Cooper [2], which we use in Sections 4 and 5.

Let D and D' be P. L. embedded bicollared disks in S^3, such that $D \cap D'$ has only clasp singularities. Let $\{\gamma_1, \ldots, \gamma_h\}$ be a basis of the free abelian group $H_1(D \cup D')$. We define two matrices A and B as follows. Let u_i be a 1-cycle representing γ_i, such that $u_i \cap (D \cap D')$ has a neighbourhood in S^3 of the form as shown in Fig. 10.

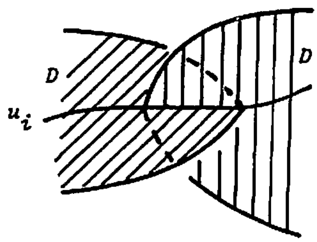


Fig. 10

Then, define $A=(lk(u_i^{--}, u_j))$ and $B=(lk(u_i^{-+}, u_j))$, where u_i^{--} (resp. u_i^{-+}) is the 1-cycle in S^3 obtained by lifting u_i off $D \cup D'$ in the negative normal direction off D and in the negative (resp. positive) normal direction off D'. The following is a special case of Theorem 2.1 of Cooper [2].

PROPOSITION 5. *The Alexander polynomial of the link* $L=\partial D \cup \partial D'$ *is* $det(xyA+A^T-xB-yB^T)$.

4. A class of knots containing 3-strand pretzel knots of odd type

For a 3-tuple of integers (r_1, r_2, r_3), let us consider an oriented link $L(r_1, r_2, r_3)=K_1 \cup K_2$ as illustrated in Fig. 11.

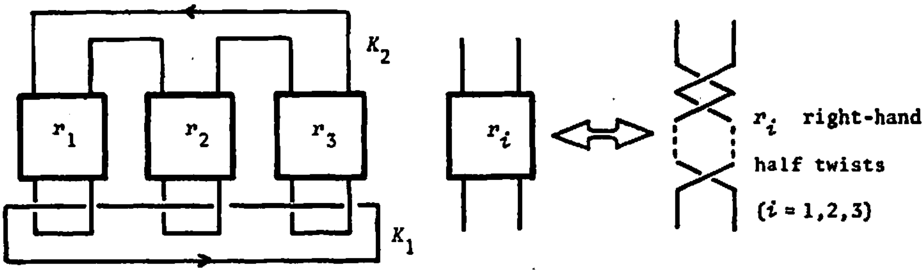


Fig. 11

Here, we assume that $(r_i, r_j) \neq (0, -1)$ for any i, j $(1 \leq i, j \leq 3)$. (If $(r_i, r_j) = (0, -1)$ for some i, j, then $L(r_1, r_2, r_3)$ is the Hopf link.) For a positive integer n $(n \geq 2)$ relatively prime to the linking number $\lambda = lk(K_1, K_2)$, let $K_n(r_1, r_2, r_3)$ be the periodic knot $C_n(L(r_1, r_2, r_3))$ generated by the link $L(r_1, r_2, r_3)$. (Recall the definition of $C_n(\ )$ given in Section 1.) In perticular, $K_2(r_1, r_2, r_3)$ is the pretzel knot of type $(2r_1+1, 2r_2+1, 2r_3+1)$.

THEOREM 1. *$K_n(r_1, r_2, r_3)$ has property P.*

PROOF. Since $L(r_1, r_2, r_3)$ is equivalent to $L(-r_1-1, -r_2-1, -r_3-1)$ and $L(r_{\sigma(1)}, r_{\sigma(2)}, r_{\sigma(3)})$ for any permutation σ on $\{1, 2, 3\}$, we may assume that $L(r_1, r_2, r_3)$ is of one of the following two types.

Type 1. $L(2l_1, 2l_2, 2l_3)$

Type 2. $L(2l_1, 2l_2, 2l_3+1)$ $\quad ((l_i, l_3) \neq (0, -1)$ for each $i = 1, 2)$

If $L(r_1, r_2, r_3)$ is of *Type* 1 (resp. *Type* 2), then the linking number λ is 3 (resp. 1). Let $\Delta(x, y)$ be the Alexander polynomial of $L(r_1, r_2, r_3)$. Then, by Propositions 2 and 4, we have only to prove that $deg\, \Delta(t, t^{k\lambda}) > \lambda - 1$ for each integer k $(|k| \geq 2)$.

To calculate $\Delta(x, y)$, let us consider bicollared disks D and D' in S^3 with $\partial D = K_1$ and $\partial D' = K_2$ as illustrated in Fig. 12. Choose 1-cycles u_1 and u_2, which form a basis of $H_1(D \cup D')$, as illustrated in Fig. 12.

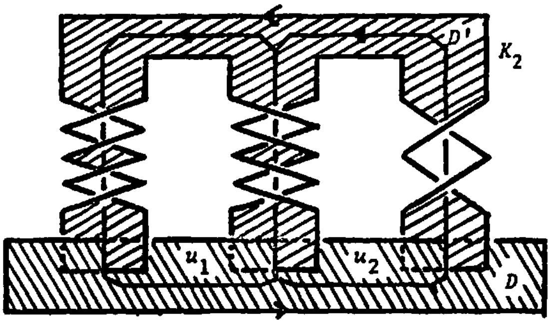


Fig. 12

Then, by Proposition 5, $\Delta(x, y) = det(xyA + A^T - xB - yB^T)$, where the matrices A and B are given as follows;

$$Type\ 1;\quad A = \begin{pmatrix} l_1+l_2 & -l_2 \\ -l_2 & l_2+l_3 \end{pmatrix},\quad B = \begin{pmatrix} l_1+l_2 & -l_2-1 \\ -l_2 & l_2+l_3+1 \end{pmatrix},$$

$$Type\ 2;\quad A = \begin{pmatrix} l_1+l_2 & -l_2 \\ -l_2 & l_2+l_3+1 \end{pmatrix},\quad B = \begin{pmatrix} l_1+l_2+1 & -l_2 \\ -l_2-1 & l_2+l_3+1 \end{pmatrix}.$$

Hence, we have $\Delta(x, y) = \sum_{1\le i,j\le 3} w_{ij}x^{i-1}y^{j-1}$, where the coefficient matrix $(w_{ij})_{1\le i,j\le 3}$ is given as follows.

$$\textit{Type } 1;\quad \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & \beta & -\alpha-2\beta & \alpha+\beta+1 \\ y & -\alpha-2\beta & 2\alpha+4\beta+1 & -\alpha-2\beta \\ y^2 & \alpha+\beta+1 & -\alpha-2\beta & \beta \end{array},$$

$$\textit{Type } 2;\quad \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & \beta+l_1+l_2 & -\alpha-2\beta-1-l_1-l_2 & \alpha+\beta+1 \\ y & -\alpha-2\beta-1-l_1-l_2 & 2\alpha+4\beta+1+2(l_1+l_2) & -\alpha-2\beta-1-l_1-l_2 \\ y^2 & \alpha+\beta+1 & -\alpha-2\beta-1-l_1-l_2 & \beta+l_1+l_2, \end{array}$$

where $\alpha=l_1+l_2+l_3$, $\beta=l_1l_2+l_2l_3+l_3l_1$.

Now, we prove that $deg\ \Delta(t, t^{k\lambda})>\lambda-1$ for any integer k ($|k|\ge 2$).

Type 1: It is clear that $(\alpha+\beta+1, -\alpha-2\beta, \beta)\ne(0, 0, 0)$. Hence, we have

$$\begin{aligned} deg\ \Delta(t, t^{k\lambda}) &= deg\ \Delta(t, t^{3k}) \\ &\ge \begin{cases} 6k-2 & (k\ge 2) \\ 0-(6k+2) & (k\le -2) \end{cases} \\ &> 2 = \lambda - 1 \end{aligned}$$

Type 2: It can be seen that $(\alpha+\beta+1, -\alpha-2\beta-1-l_1-l_2)=(0, 0)$ (resp. $(-\alpha-2\beta-1-l_1-l_2, \beta+l_1+l_2)=(0, 0)$), iff $l_3=-1$ and $l_1l_2=0$. But this does not occur by the assumption. Hence, for any k ($|k|\ge 2$), we have

$$deg\ \Delta(t, t^{k\lambda}) = deg\ \Delta(t, t^k) \ge |(2k+1)-1| > 0 = \lambda - 1.$$

This completes the proof.

5. A class of knots containing 2-bridge knots

Let $L(2p, q)=K_1\cup K_2$ be an oriented 2-bridge link of type $(2p, q)$, where $1\le q<2p$ and $g.c.d.\ (2p, q)=1$ (see Fig. 13). Here, we assume that $p\ne 1$. ($L(2, 1)$ is the Hopf link.) For a positive integer n ($n\ge 2$) relatively prime to the linking number $\lambda=lk(K_1, K_2)$, let $K_n(p, q)$ be the periodic knot $C_n(L(2p, q))$ generated by the link $L(2p, q)$. In particular, $K_2(p, q)$ is a 2-bridge knot of type (p, q).

THEOREM 2. *$K_n(p, q)$ has property P.*

PROOF. $\frac{2p}{q}$ has the following continued fraction;

$$\frac{2p}{q} = 2b_1 + \frac{1}{-2b_2} + \frac{1}{2b_3} + \cdots\cdots + \frac{1}{-2b_{m-1}} + \frac{1}{2b_m},$$

where b_i is a non-zero integer for each i $(1 \leq i \leq m)$, and m is an odd integer. Then $L(2p, q)$ is equivalent to the link as illustrated in Fig. 13.

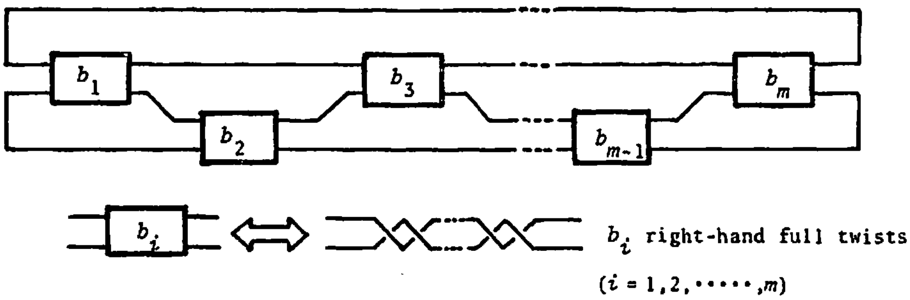
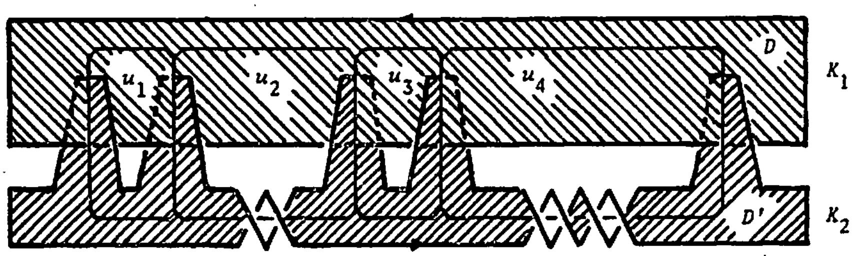


Fig. 13

We may assume that the linking number λ is positive. Let $\Delta(x, y)$ be the Alexander polynomial of $L(2p, q)$. Then, by Propositions 2 and 4, we have only to prove that $deg\ \Delta(t, t^{k\lambda}) > \lambda - 1$ for each k $(|k| \geq 2)$. To calculate the polynomial, let us consider bicollared disks D and D' with $\partial D = K_1$ and $\partial D' = K_2$ as illustrated in Fig. 14. $D \cap D'$ consists of $|b_1| + |b_3| + \cdots + |b_m|$ clasp singularities. Choose 1-cycles $u_1, u_2, \ldots, u_s$ which form a basis of $H_1(D \cup D')$ as illustrated in Fig. 14, where $s = |b_1| + |b_3| + \cdots + |b_m| - 1$. Note that $s \geq 1$, since $p \neq 1$.

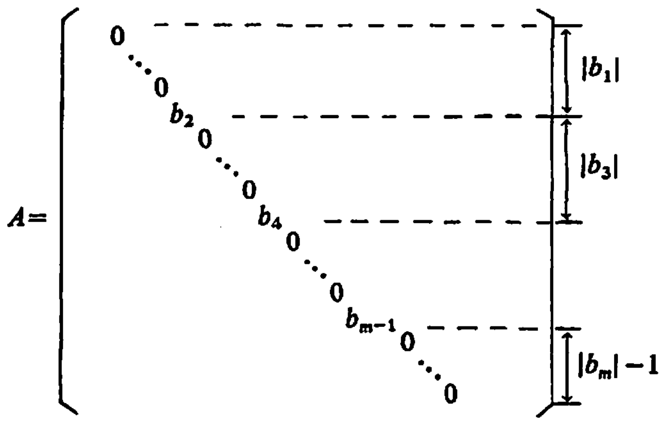


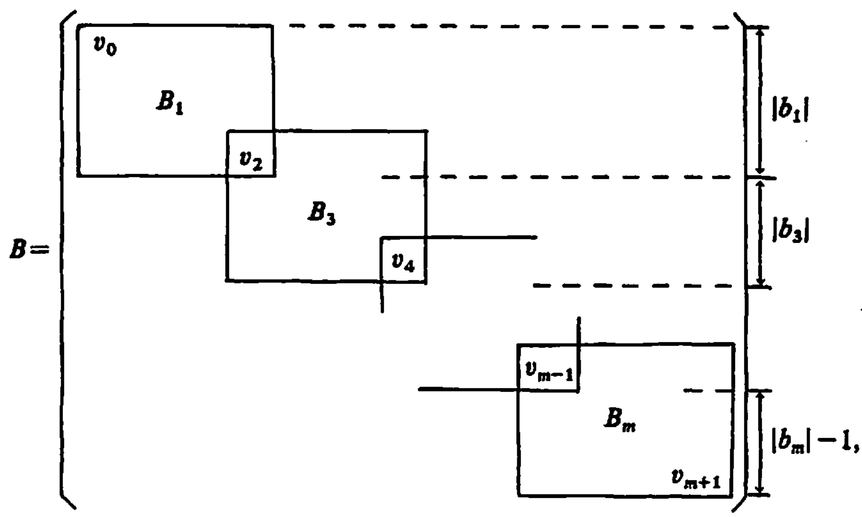
Shaded sides of D and D' are negative sides.

Fig. 14

Let $V = t^{k\lambda+1}A + A^T - tB - t^{k\lambda}B^T$, where A and B are matrices defined in Section 3. Then, by Proposition 5, $\Delta(t, t^{k\lambda}) = det\ V$.

The matrices A and B are given as follows;





where B_i and v_j are given as follows ($i=1, 3, 5, \ldots, m$, $j=0, 2, 4, \ldots, m+1$).

$$B_i=\begin{cases}\begin{pmatrix} v_{i-1} & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -1 & 1 \\ & & & & & v_{i+1}\end{pmatrix} & (b_i>0)\\[2ex] \begin{pmatrix} v_{i-1} & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & -1 \quad v_{i+1}\end{pmatrix} & (b_i<0)\end{cases}$$

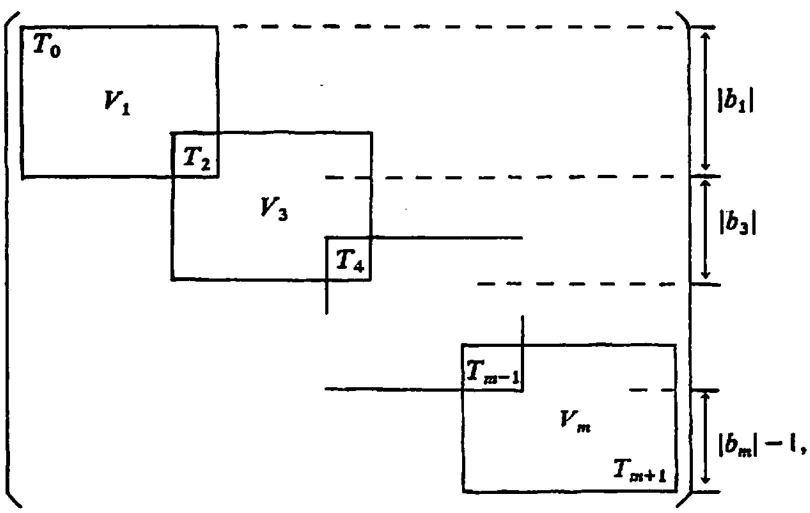
$$v_0=\begin{cases}-1 & (b_1>0)\\ 1 & (b_1<0)\end{cases}$$

$$v_j=b_j+\varepsilon_j/2 \quad \text{with} \quad \varepsilon_j=-\{b_{j-1}/|b_{j-1}|+b_{j+1}/|b_{j+1}|\} \quad (j=2, 4, \dots, m-1)$$

$$v_{m+1}=\begin{cases}-1 & (b_m>0)\\ 1 & (b_m<0)\end{cases}$$

Therefore the matrix $V=t^{k\lambda+1}A+A^{\mathrm{T}}-tB-t^{k\lambda}B^{\mathrm{T}}$ is of the following form.

$V=$



where V_i and T_j are given as follows ($i=1, 3, 5, \dots, m$, $j=0, 2, 4, \dots, m+1$).

$$V_l = \begin{cases} \begin{pmatrix} T_{l-1} & -t & & & \\ -t^{k\lambda} & t^{k\lambda}+t & -t & & \\ & \ddots & \ddots & \ddots & \\ & & -t^{k\lambda} & t^{k\lambda}+t & -t \\ & & & -t^{k\lambda} & T_{l+1} \end{pmatrix} & (b_l>0) \\ \begin{pmatrix} T_{l-1} & t^{k\lambda} & & & \\ t & -t^{k\lambda}-t & t^{k\lambda} & & \\ & \ddots & \ddots & \ddots & \\ & & t & -t^{k\lambda}-t & t^{k\lambda} \\ & & & t & T_{l+1} \end{pmatrix} & (b_l<0) \end{cases}$$

$$T_0 = \begin{cases} t^{k\lambda} + t & (b_1>0) \\ -t^{k\lambda} - t & (b_1<0) \end{cases}$$

$$T_j = b_j(t^{k\lambda+1}+1) - (b_j+\varepsilon_j/2)(t^{k\lambda}+t) \qquad (j=2, 4,\dots, m-1)$$

$$T_{m+1} = \begin{cases} t^{k\lambda} + t & (b_m>0) \\ -t^{k\lambda} - t & (b_m<0) \end{cases}$$

Let W_i $(1\leq i\leq s)$ be the submatrix of V consisting of (ν, μ) entries of V with $\nu, \mu> s-i$. Define $d_i(t)=det\, W_i$ $(1\leq i\leq s)$, and $d_0(t)=1$. Especially, $d_s(t)=det\, V=\Delta(t, t^{k\lambda})$.

LEMMA 1. *For each integer i $(1\leq i\leq s-1)$, the following equation holds.*

$$d_{i+1}(t) = F_{s-i}(t)d_i(t) - t^{k\lambda+1}d_{i-1}(t).$$

Here, $F_j(t)$ is the (j, j) entry of V.

PROOF. By expanding the first column of W_i, we obtain the equation immediately.

For a Laurent polynomial $f(t)$, let $Max(f(t))$ (resp. $Min(f(t))$) be the maximal (resp. minimal) t-power of any term of $f(t)$.

LEMMA 2. *For each integer i $(1\leq i\leq s)$, we have the followings.*

(1) *If* $k\geq 2$, $\quad Max(d_i(t))\geq Max(d_{i-1}(t)) + k\lambda \quad \cdots\cdots(\alpha_i)$,

$\qquad Min(d_i(t))\leq Min(d_{i-1}(t)) + 1 \quad \cdots\cdots(\beta_i)$.

(2) *If* $k\leq -2$, $\quad Max(d_i(t))\geq Max(d_{i-1}(t)) \quad \cdots\cdots(\gamma_i)$,

$\qquad Min(d_i(t))\leq Min(d_{i-1}(t)) + k\lambda + 1 \quad \cdots\cdots(\delta_i)$.

PROOF. (1) $k\geq 2$: Note that $Max(F_i(t))\geq k\lambda$ and $Min(F_i(t))\leq 1$, for each i $(1\leq i\leq s)$. We prove the inequality (α_i) inductively. Since $Max(d_1(t))=Max(F_s(t))\geq k\lambda$, (α_1) holds. Suppose that (α_j) holds for some j $(1\leq j\leq s-1)$.

Then
$$\begin{aligned} Max(F_{s-j}(t)d_j(t)) &= Max(F_{s-j}(t)) + Max(d_j(t)) \\ &\geq k\lambda + (Max(d_{j-1}(t))+k\lambda) \\ &> (k\lambda+1) + Max(d_{j-1}(t)) \\ &= Max(t^{k\lambda+1}d_{j-1}(t)). \end{aligned}$$

Hence, by Lemma 1, $Max(d_{j+1}(t))=Max(F_{s-j}(t)d_j(t))\geq Max(d_j(t))+k\lambda$, and (α_{j+1}) holds. Therefore (α_i) holds for any i $(1\leq i\leq s)$. Next, we prove (β_i) inductively. Since $Min(d_1(t))=Min(F_s(t))\leq 1$, (β_1) holds. Suppose that (β_j) holds for some j $(1\leq j\leq s-1)$. Then

$$\begin{aligned} Min(F_{s-j}(t)d_j(t)) &= Min(F_{s-j}(t)) + Min(d_j(t)) \\ &\leq 1 + (Min(d_{j-1}(t))+1) \\ &< (k\lambda+1) + Min(d_{j-1}(t)) \\ &\leq Min(t^{k\lambda+1}d_{j-1}(t)). \end{aligned}$$

Hence, by Lemma 1, $Min(d_{j+1}(t))=Min(F_{s-j}(t)d_j(t))\leq Min(d_j(t))+1$, and (β_{j+1}) holds. Therefore (β_i) holds for any i $(1\leq i\leq s)$.

(2) $k\leq -2$: Note that $Max(F_i(t))\geq 0$ and $Min(F_i(t))\leq k\lambda+1$, for each i $(1\leq i\leq s)$. Then, by a similar argument as the above, we can prove the inequalities (γ_i) and (δ_i) $(1\leq i\leq s)$.

From the above lemma, we have $deg(d_i(t))>deg(d_{i-1}(t))$, for any i $(2\leq i\leq s)$, and $deg(d_1(t))>\lambda-1$. Therefore,
$$deg\,\Delta(t, t^{k\lambda}) = deg(d_s(t)) > deg(d_{s-1}(t)) > \cdots > deg(d_1(t)) > \lambda - 1.$$
This completes the proof of Theorem 2.

6. Even pretzel knots

Let $K(p, q, 2r)$ be an even prezel knot. (p and q are odd integers.) Riley [17] proved that, if $p+q\neq 0$, then $K(p, q, 2r)$ has Property PP. So, we consider $K(p, -p, 2r)$. Note that $K(p, -p, 2r)$ is nontrivial, iff $|p|\neq 1$.

THEOREM 3. *$K(p, -p, 2r)$ (p: odd, $|p|\neq 1$) has Property P_{2k+1} for any integer k.*

PROOF. Since $K(p, -p, 2r)=K(-p, p, -2r)$, we may assume that $p=4p'+1$. Let O and J be a trivial knot and an arc in S^3 as illustrated in Fig. 15. Then $K(p, -p, 2r)=I(J)$.

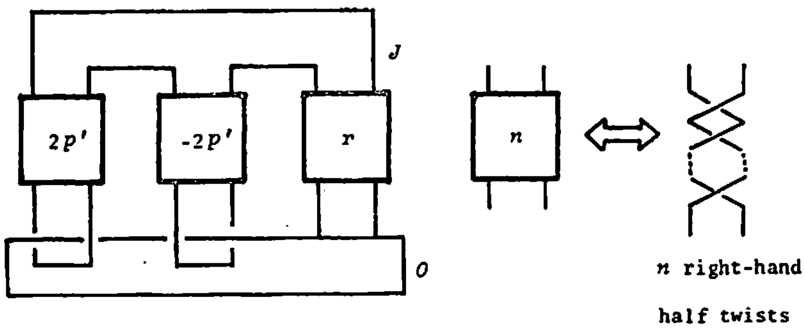


Fig. 15

By Proposition 3, we have only to prove that the knot $F_{2k+1}(J)$ is nontrivial. Recall that $F_{2k+1}(J)=W_k(L_1(J))$. So, by Proposition 4, the Alexander polynomial $\Delta^{(2k+1)}(t)$ of $F_{2k+1}(t)$ is given by

$$\Delta^{(2k+1)}(t) = \Delta^{(1)}(t) + kA_1(t),$$

where $A_1(t)$ is the $A\tau$-polynomial of the link $L_1(J)$. By direct calculation, we have

$$\Delta^{(1)}(t) = \begin{cases} [4p'^2 + 2p' + 1, \ - p'^2 - p', \ - 2p'^2 - p', \ p'^2 + p' & (r\text{: odd}) \\ [4p'^2 + 2p' + 1, \ - p'^2, \ - 2p'^2 - p', \ p'^2 & (r\text{: even}), \end{cases}$$

and $A_1(t)=0$.
Since $|p|=|4p'+1|\neq 1$ by the assumption, we have $p'\neq 0$. Therefore $\Delta^{(2k+1)}(t)\neq 1$. This completes the proof.

REMARK. Since the link $L_0(J)$ is slice in the strong sense, the $A\tau$-polynomial of $L_0(J)$ is zero (see [9]). So, the Alexander polynomial of the knot $F_{2k}(J)$ is 1.

7. Knots with 9 crossings or less

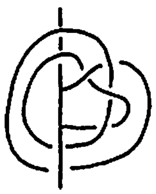
Riley [17] proved that all knots with 9 crossings or less have Preperty PP except 8_{10}, 8_{17}, and 9_n for $n=24$, 29, 32, 33, 34, 38, 39, 41, 46, 47, and 49. In this section, we apply our method to them, and prove that all of them except 8_{17}, 9_{32}, and 9_{33} have Property P.

First, we study 8_{10} and 9_{24} from a different point of view. 8_{10} and 9_{24} are "ribbon concordant" to 3_1 and 4_1 respectively, and therefore, there are epimorphisms from the knot groups of 8_{10} and 9_{24} to those of 3_1 and 4_1 respectively, which carry meridians to meridians and longitudes to longitudes. Therefore, it follows that 8_{10} and 9_{24} have Property P, since 3_1 and 4_1 do. (Recently, Osborn [16] proved that 8_{10} has Property P by a different method.)

Next, we use the method of Section 1. Among the knots in consideration, only 9_{41}, 9_{46}, 9_{47}, and 9_{49} are periodic (see [15]). 9_{46} is a pretzel knot of type $(3, 3, -3)$; so, by Theorem 1, 9_{46} has Property P. 9_{41}, 9_{47}, and 9_{49} belong to the class of knots considered in Section 5. In fact, $9_{41} \cong K_3(9, 5)$, $9_{47} \cong K_3(8, 3)$, and $9_{49} \cong K_3(7, 3)$ (see [4]). Thus, by Theorem 2, they have Property P.

For the remaining knots 9_{29}, 9_{34}, 9_{38}, and 9_{39}, which are strongly invertible, we use the method of Section 2. The following is a list of the corresponding θ-curves $O \cup J$, the $A\tau$-polynomials $A_0(t)$ and $A_1(t)$ of the links $L_0(J)$ and $L_1(J)$, and the Alexander polynomials $\Delta^{(1)}(t)$ of the knots $F_1(J)$ (cf. Example 5).

9_{29}



$A_0(t) = [2, -1, -1, 1$
$\Delta^{(1)}(t) = [3, 0, -3, 0, 2, 0, -1,$
$A_1(t) = \pm [6, -2, -4, 3, 2, -2, -1, 1$

9_{34}

$A_0(t) = [0, 1, -1, -1, 1$
$\Delta^{(1)}(t) = [1, 0, 1, -1, -2, 1, 1$
$A_1(t) = \pm [0, -1, 2, 1, -3, 0, 1$

9_{38}

$A_0(t) = [4, 0, -2$
$\Delta^{(1)}(t) = [-1, 1, -1, 1, -1, 1, 0, -3, 2, 2, -1, -2, 1$
$A_1(t) = \pm [-2, 2, -2, 2, -2, 1, 3, -5, 0, 3, 1, -3, 1$

9_{39}

$A_0(t) = [-4, 0, 2$
$\Delta^{(1)}(t) = [1, -1, 1, -1, 0, 1, -2, 1, 1, 0, -1$
$A_1(t) = \pm [2, -2, 2, -2, 1, 1, -3, 3, 0, -1, -1, 1$

From the above list and Propositions 3 and 4, it follows that 9_{29}, 9_{34}, 9_{38}, and 9_{39} have Property P.

8. Final Remark

Litherland [10, 11] proved that, for a 2-component link $L = O \cup K$ in S^3 with O a trivial knot, if one of the following conditions holds, then the exterior of the knot $W_k(L)$ in $M(K, 1/k)$ is not a homotopy solid torus.

(1) $|lk(O, K)| \geq 3$ and $k \neq 0$.
(2) $|lk(O, K)| = 2$ and $|k| \geq 2$.
(3) $|lk(O, K)| = 1$, $wr(O, K) \geq 2$, and $|k| \geq 6$. (Here $wr(O, K)$ is the minimum number of intersections of K with a disk bounded by O.)

Hence, the following holds by Proposition 2.

THEOREM 4. *$C_n(L)$ has Property P, if one of the following conditions holds.*
(1) $|lk(O, K)| \geq 2$.
(2) $|lk(O, K)| = 1$, *$n \geq 6$, and L is not a Hopf link.*

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