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Dirichlet problem and Green's formulas on trees

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ABSTRACT. The Dirichlet problem and the construction of superharmonic functions with point harmonic singularities are two of the basic problems in potential theory. In this article, we study these problems in the context of discrete potential theory, which leads to the consideration of Green's formulas and flux on a Cartier tree.

1. Introduction

In the study of potential theory, classical as well as axiomatic, two of the basic problems are the Dirichlet problem and the construction of potentials with point harmonic support. In this note, we consider these two problems in the context of a Cartier tree (which is an infinite connected graph, locally finite and without loops).

Then we present a version of the Green's formula in a tree, which corresponds to the equality

$$\int\int_{\Omega}(f\Delta g - g\Delta f)d\sigma = \int_{\partial\Omega}\left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)dS,$$

where Ω is a bounded open set in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ of class C^1, f and g are C^2-functions on a neighbourhood of $\bar{\Omega}$, and $\frac{\partial}{\partial n}$ denotes the outer normal derivative. Such formulas are known in the context of discrete analysis on graphs (see H. Urakawa [6]). To conclude, we derive some consequences from this formula in a tree (as in the classical potential theory), including the rôle of flux.

2. Preliminaries

By a Cartier tree T [3, p. 208], we mean a countably infinite set of vertices, some of which are pairwise joined by edges (T *is an infinite graph*); if two vertices x and y are joined by an edge, x and y are said to be neighbours, denoted by $x \sim y$; a vertex can have only a finite number of neighbours (T

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is locally finite); if u and v are any two vertices, either they are neighbours or there exist a finite number of vertices $x_1, \dots, x_{n-1}$ such that in the path $\{u = x_0, x_1, \dots, x_{n-1}, x_n = v\}$, $x_i \sim x_{i+1}$ for $0 \le i \le n-1$ (*T is connected*); finally, if x and y are neighbours, there is no path $\{x = s_0, s_1, \dots, s_{n-1}, s_n = y\}$, $n \ge 2$, with distinct vertices such that $s_i \sim s_{i+1}$ for $1 \le i \le n-1$ (*T has no loops*); a vertex x_0 is said to be *terminal* if it has only one neighbour in T.

Given a set E of vertices in T, $x \in E$ is said to be an *interior point* of E if x is not a terminal vertex and if all the neighbours of x in T are also in E; we denote by $\mathring{E}$, the collection of all the interior points of E. Let us put $\partial E = E \backslash \mathring{E}$. On a tree T, a *transition probability* is assumed to be given: that is, with any two vertices x and y is associated a real number $p(x, y) \ge 0$ such that (i) $p(x, y) > 0$ if and only if x and y are neighbours; and (ii) $\sum_{x \sim y} p(x, y) = 1$ for any x in T.

Finally, if u is any real-valued function on T, the Laplacian Δu of u at a vertex x is defined as $\Delta u(x) = \sum\limits_{x \sim x_i} p(x, x_i) u(x_i) - u(x)$; since $p(x, x_i) = 0$ if x and x_i are not neighbours, we simply write $\Delta u(x) = \sum\limits_{y \in T} p(x, y)[u(y) - u(x)]$. The function u is said to be harmonic (respectively superharmonic) at a vertex x, if x is not terminal and if $\Delta u(x) = 0$ (respectively $\Delta u(x) \le 0$).

Theorem 1. *Let e be a vertex in T. Then there exists a function $g_e(x)$ on T such that $\Delta g_e(x) = -\delta_e(x)$, the Dirac measure δ_e.*

Proof. For any x in T, we shall denote by $|x| = d(e, x)$ the number of edges in the unique geodesic path from e to x.

Define a function u with values $u(e) = 1$ and for x satisfying $|x| = 1$, take $u(x) = 0$ if there is some nonterminal vertex $y \sim x$ with $|y| = 2$, otherwise take $u(x) = 1$. Let y be a vertex with $|y| = 2$. Let $x \sim y$ for which $|x| = 1$. Now x can have neighbours other than y and e; denote them by $y_1, \dots, y_i$. In the set $A = \{y, y_1, \dots, y_i\}$, some may be terminal vertices; denote this subset of terminal vertices by A_1. Let $A_2 = A \backslash A_1$. If $A_2 \ne \phi$, define $u = 0$ on A_1 and $u =$ a constant α on A_2, such that $\Delta u(x) = 0$. For this to happen, we have to choose α so that $p(x, e) + \alpha \sum\limits_{z \in A_2} p(x, z) = 0$. If $A_2 = \phi$, take $u = 1$ on $A_1 = A$, so that $u(x) = 1$ and $\Delta u(x) = 0$.

Suppose z is a vertex such that $|z| = 2$ and z is different from $y, y_1, \dots, y_i$. Then the above method can be repeated to define $u(z)$, so that if $x' \sim z$ and $|x'| = 1$, then $\Delta u(x') = 0$. Thus proceeding, we extend the definition of $u(x)$ to all $|x| = 2$ in such a way that $\Delta u(x) = 0$ for all $|x| = 1$.

Continuing this process, we define a function u on T such that $\Delta u(x) = 0$ for all $|x| \ge 1$; moreover, if $u(x) = 1$ for all $|x| = 1$, there can not be any vertex y such that $|y| \ge 3$, which is a contradiction. Hence $u(x) = 0$ for some

$|x| = 1$, which means that $u(e) < 0$. Define $g_e(x) = \frac{u(x)}{-\Delta u(e)}$. Then $g_e(x)$ is superharmonic on T, $\Delta g_e(x) = 0$ if $x \neq e$ and $\Delta g_e(e) = -1$. Hence the theorem is proved. $\square$

Remark. $g_e(x)$ may or may not be lower bounded on T.

Theorem 2 (Dirichlet problem). *Let B be a finite set of connected vertices in T. Suppose $f(x)$ is a function defined on ∂B. Then there exists a unique function $h(x)$ on B such that $h(x)$ is harmonic on $\mathring{B}$ and $h(x) = f(x)$ for $x \in \partial B$.*

Proof. Since we can consider the functions f^+ and f^- separately and solve the Dirichlet problem, we shall assume that $f \geq 0$, without any loss of generality.

Choose two constants α and β such that $0 \leq \beta \leq f(x) \leq \alpha$ for $x \in \partial B$. Let

$$s(x) = \begin{cases} \alpha & \text{if } x \in \mathring{B} \\ f(x) & \text{if } x \in \partial B, \end{cases}$$

and

$$t(x) = \begin{cases} \beta & \text{if } x \in \mathring{B} \\ f(x) & \text{if } x \in \partial B. \end{cases}$$

Then, $s(x)$ and $t(x)$ are defined on B, $s(x)$ is superharmonic on $\mathring{B}$, $t(x)$ is subharmonic on $\mathring{B}$, and $t(x) \leq s(x)$ for $x \in B$.

Let $\mathscr{F}$ be the family of all subharmonic functions u on $\mathring{B}$ such that $u(x) \leq s(x)$ on B. Note that $\mathscr{F}$ is an increasingly filtered family. Let $h(x) = \sup_{u \in \mathscr{F}} u(x)$. Since $t(x) \leq h(x) \leq s(x)$ on ∂B, $h(x) = f(x)$ on ∂B. We shall prove that $h(x)$ is harmonic on $\mathring{B}$.

Let $z \in \mathring{B}$. Let $\{z_i\}$ be the complete set of neighbours of z. Let $u \in \mathscr{F}$. Then $u(z) \leq \sum_{z \sim z_i} p(z, z_i) u(z_i) \leq \sum_{z \sim z_i} p(z, z_i) s(z_i) \leq s(z)$. Define

$$u_1(x) = \begin{cases} u(x) & \text{if } x \neq z \text{ and } x \in B \\ \sum_{z \sim z_i} p(z, z_i) u(z_i) & \text{if } x = z. \end{cases}$$

Then u_1 is subharmonic on $\mathring{B}$, $u(x) \leq u_1(x) \leq s(x)$ on B and $u_1(x)$ is harmonic at $x = z$. Hence $u_1 \in \mathscr{F}$. This modification is possible with respect to each $u \in \mathscr{F}$, for a fixed z. Let us put for each $u \in \mathscr{F}$, and for a fixed z, the modified function u_1 as indicated above in a subclass $\mathscr{F}'$. Clearly $\mathscr{F}' \subset \mathscr{F}$. Hence $\sup_{\mathscr{F}} u \geq \sup_{\mathscr{F}'} u_1$. But at z, we have $u(z) = u_1(z)$. Consequently $h(z) = \sup_{u \in \mathscr{F}} u(z) = \sup_{u_1 \in \mathscr{F}'} u_1(z)$. Since $\mathscr{F}$ is increasingly filtered, h is harmonic at the vertex z.

Since z is an arbitrary vertex with the sole restriction that $z \in \mathring{B}$, $h(x)$ is harmonic on $\mathring{B}$; also, since $h(x) = f(x)$ for $x \in \partial B$, h is the desired Dirichlet solution on B.

Uniqueness of the solution: To prove the uniqueness of h, it is enough to show that if $H(x)$ is defined on B, harmonic on $\mathring{B}$, and $H(x) = 0$ if $x \in \partial B$, then $H \equiv 0$. This follows immediately from the maximum principle for harmonic functions, since B is connected. □

Remarks.

1. The above proof is based on potential theoretic methods. An alternate method of solving the Dirichlet problem for a special type of finite subset of T is given in Berenstein et al. [1, p. 461], using the hitting distribution of the stochastic process generated by the transition probability structure of T.
2. The above proof of the Dirichlet solution goes through in unbounded sets also. For example:
   Let E be a (not necessarily finite) set of vertices. Let f be a bounded function on ∂E. Then there exists a bounded function h on E such that h is harmonic on $\mathring{E}$ and $h = f$ on ∂E. However, to prove the uniqueness of h we need E to be finite, $\mathring{E}$ is connected and a vertex on ∂E has a neighbour in $\mathring{E}$.

Now we shall define the notions of the outer and the inner normal derivatives in T. Let E be a set of vertices in T; Let $E^\star = E \bigcup$ (neighbours of vertices in E); that is, $x \in E^\star$ if and only if either $x \in E$ or x is a neighbour of a vertex in E; let $\partial E = E \backslash \mathring{E}$.

Let $x \in \partial E$. Let $\{x_1, \dots, x_i\}$ be the neighbours of x in E and $\{y_1, \dots, y_j\}$ be the neighbours of x outside E.

Definition 1. Suppose u is defined on $E^\star$. Then *the outer normal derivative* of u at a point $x \in \partial E$ is

$$\frac{\partial u}{\partial n^+}(x) = \sum_{y_\alpha \sim x, y_\alpha \notin E} p(x, y_\alpha)[u(y_\alpha) - u(x)].$$

Definition 2. Suppose v is defined on E. Then *the inner normal derivative* of v at a point $x \in \partial E$ is

$$\frac{\partial v}{\partial n^-}(x) = \sum_{x_\beta \sim x, x_\beta \in E} p(x, x_\beta)[v(x_\beta) - v(x)].$$

Note. If u is defined on $E^\star$ and if $x \in \partial E$, then

$$\begin{aligned}\frac{\partial u}{\partial n^+}(x) + \frac{\partial u}{\partial n^-}(x) &= \sum_{z \sim x} p(x, z)[u(z) - u(x)] \\ &= \sum_{z \sim x} p(x, z)u(z) - u(x) = \Delta u(x).\end{aligned}$$

3. Green's formulas

We shall prove now a version of the Green's first formula for a tree, motivated by a result of Duffin [4, Lemma 1] in the discrete situation of lattice points in $\mathbf{R}^3$, of the form (a,b,c) where a, b, and c take on the values $0, \mp 1, \mp 2, \ldots$; for a real-valued function u on the lattice points, Duffin introduces the Laplace operator D by defining

$$Du(a,b,c) = u(a+1,b,c) + u(a-1,b,c) + u(a,b+1,c) + u(a,b-1,c) \\ + u(a,b,c+1) + u(a,b,c-1) - 6u(a,b,c).$$

Such a formula in the context of discrete analysis on graphs is also known, see Urakawa [6]. Here is a version of this formula in the framework of a tree.

Fix a vertex e in T. For a vertex x, let $\{e, x_1, \ldots, x_n, x\}$ be a path joining e and x. Write $\phi(x) = \frac{p(e,x_1)p(x_1,x_2)\ldots p(x_n,x)}{p(x,x_n)p(x_n,x_{n-1})\ldots p(x_1,e)}$; take $\phi(e) = 1$. Since T has no loops, it is easy to see that $\phi(x)$ is independent of the path chosen to join e and x. If x and y are neighbours, then $\phi(y) = \phi(x)\frac{p(x,y)}{p(y,x)}$; that is, $\phi(x)p(x,y) = \phi(y)p(y,x)$. Clearly this equality holds even when x and y are not neighbours. Thus, for any pair of vertices x and y in T, if we define $\psi(x,y) = \phi(x)p(x,y)$, then $\psi(x,y) = \psi(y,x) \geq 0$, and $\psi(x,y) = 0$ if and only if x and y are not neighbours. Let now E be a (not necessarily finite) set of vertices in T. Let u and v be two real-valued functions defined on E, such that $\sum_{x,y \in E} \psi(x,y)u(x)[v(y) - v(x)]$ is absolutely convergent. Then write

$$(u,v)_E = \sum_{x,y \in E} \psi(x,y)u(x)[v(y) - v(x)].$$

In this sum, corresponding to a pair of points x and y in E, we have two terms $\psi(x,y)u(x)[v(y) - v(x)]$ and $\psi(y,x)u(y)[v(x) - v(y)]$. Their sum is $-\psi(x,y)[u(y) - u(x)][v(y) - v(x)]$, since $\psi(x,y) = \psi(y,x)$. Consequently, rearranging the terms in $(u,v)_E$ we have

$$(u,v)_E = -\frac{1}{2}\sum_{x,y \in E} \psi(x,y)[u(y) - u(x)][v(y) - v(x)].$$

This implies that whenever $(u,v)_E$ and $(v,u)_E$ are defined, they are equal. Remark that in the case of E being a finite set, for any real functions u and v on E, $(u,v)_E$ is always well-defined in the above form.

Theorem 3. *Let E be a finite set of vertices. Suppose u and v are two real-valued functions defined on E. Then*

$$\sum_{x \in \mathring{E}} \phi(x)u(x)\Delta v(x) - (u,v)_E = -\sum_{s \in \partial E} \phi(s)u(s)\frac{\partial v}{\partial n^-}(s).$$

Proof. Extend u and v arbitrarily outside E. Then,

$$\begin{aligned}\sum_{x\in E}\phi(x)u(x)\Delta v(x) &= \sum_{x\in E}\phi(x)u(x)\sum_{y\in T}p(x,y)[v(y)-v(x)]\\ &= \sum_{x\in E}\sum_{y\in T}\psi(x,y)u(x)[v(y)-v(x)]\\ &= \sum_{x\in E}\sum_{y\in E^\star}\psi(x,y)u(x)[v(y)-v(x)]\\ &= \sum_{x\in E}\sum_{y\in E}\psi(x,y)u(x)[v(y)-v(x)]\\ &\quad+\sum_{x\in E}\phi(x)u(x)\sum_{y\in E^\star\setminus E}p(x,y)[v(y)-v(x)]\end{aligned}$$

(using the fact that $\psi(x,y)>0$ only if $x\sim y$ in the third equality). The first double sum on the right side is $(u,v)_E$; the second sum reduces to $\sum_{x\in\partial E}\phi(x)u(x)\sum_{y\in E^\star\setminus E}p(x,y)[v(y)-v(x)]$, since $p(x,y)=0$ if $y\in E^\star\setminus E$ and $x\in\mathring{E}$. Hence,

$$\begin{aligned}\sum_{x\in E}\phi(x)u(x)\Delta v(x) &= (u,v)_E+\sum_{x\in\partial E}\phi(x)u(x)\frac{\partial v}{\partial n^+}(x)\\ &= (u,v)_E+\sum_{x\in\partial E}\phi(x)u(x)\left[\Delta v(x)-\frac{\partial v}{\partial n^-}(x)\right].\end{aligned}$$

A cancellation of the term $\sum_{x\in\partial E}\phi(x)u(x)\Delta v(x)$ on both sides leads to the theorem. □

Remark.

1. The above proof shows that Theorem 3 is valid on an infinite set E also, provided $(u,v)_E$ is defined and ∂E has only a finite number of vertices.
2. Suppose $s\in\partial E$ is a terminal vertex. Then $\frac{\partial v}{\partial n^-}(s)=0$ or $\Delta v(s)$, depending on whether the neighbour of s is outside or inside E. Hence, in the above Theorem 3, if $E=\{x:|x|\le m\}$ we have the following form

$$\sum_{|x|<m}\phi(x)u(x)\Delta v(x)-(u,v)_E=-\sum_{|s|=m}\phi(s)u(s)\frac{\partial v}{\partial n^-}(s).$$

A variant of this formula with outer normal derivatives is

$$\sum_{|x|\le m}\phi(x)u(x)\Delta v(x)-(u,v)_E=\sum_{|s|=m}\phi(s)u(s)\frac{\partial v}{\partial n^+}(s).$$

As a consequence of Theorem 3 and its proof we have

Theorem 4. *Let E be a finite set of vertices on T.*
i) *Suppose u and v are defined on E. Then*

$$\sum_{x \in \mathring{E}} \phi(x)[u(x)\Delta v(x) - v(x)\Delta u(x)]$$

$$= -\sum_{s \in \partial E} \phi(s)\left[u(s)\frac{\partial v}{\partial n^-}(s) - v(s)\frac{\partial u}{\partial n^-}(s)\right].$$

ii) *Suppose u and v are defined on $E^\star$. Then*

$$\sum_{x \in E} \phi(x)[u(x)\Delta v(x) - v(x)\Delta u(x)] = \sum_{s \in \partial E} \phi(s)\left[u(s)\frac{\partial v}{\partial n^+}(s) - v(s)\frac{\partial u}{\partial n^+}(s)\right].$$

4. Some consequences

In this section, we derive some consequences of Theorem 4.

Consequence 1. *Let E be a finite set of vertices.*
i) *Suppose u is defined on E. Then $\sum_{\mathring{E}} \phi(x)\Delta u(x) = -\sum_{\partial E} \phi(s)\frac{\partial u}{\partial n^-}(s)$.*
ii) *Suppose u is defined on $E^\star$. Then $\sum_{E} \phi(x)\Delta u(x) = \sum_{\partial E} \phi(s)\frac{\partial u}{\partial n^+}(s)$.*

Proof. In Theorem 4, take $v \equiv 1$. Note that $\Delta v \equiv 0$, and for $s \in \partial E$, $\frac{\partial v}{\partial n^+}(s) = \frac{\partial v}{\partial n^-}(s) = 0$. $\square$

Remark. Apparently the above results depend on the choice of e which is used to define $\phi(x)$. But in reality the choice of e does not play any role here. For, if $\phi'(x)$ denotes the value corresponding to another choice e', we have $\phi'(x) = \phi'(e)\phi(x)$.

We say that a function u defined on E is superharmonic (respectively harmonic) on E if $\Delta u(x) \leq 0$ (respectively $\Delta u(x) = 0$) for every $x \in \mathring{E}$.

Consequence 2. *Let $u(x)$ be a superharmonic function on a finite set E. Then u is harmonic on E if and only if $\sum_{\partial E} \phi(s)\frac{\partial u}{\partial n^-}(s) = 0$.*

Proof. If u is harmonic on E, $\Delta u \equiv 0$ on $\mathring{E}$ and hence by Theorem 4, $\sum_{\partial E} \phi(s)\frac{\partial u}{\partial n^-}(s) = 0$. Conversely, if this condition is satisfied, then $\sum_{\mathring{E}} \phi(x)\Delta u(x) = 0$. Since $\Delta u \leq 0$ also on $\mathring{E}$, $\Delta u \equiv 0$ on $\mathring{E}$. $\square$

Consequence 3. *Let $f(x)$ be a function on T such that $f = 0$ outside a finite set. Then $\sum_{T} \phi(x)\Delta f(x) = 0$.*

Proof. Let $f = 0$ outside a finite set A. Then $A^\star = A \bigcup$ (neighbours of vertices in A) is also finite. Consequently, if we measure distances from a fixed vertex e, then $d(e,x) = |x| \le m$ for some integer m and for every $x \in A^\star$. Let $E = \{x : |x| \le m+1\}$. Since $\sum_E \phi(x)\Delta f(x) = \sum_{\partial E} \phi(x)\frac{\partial f}{\partial n^+}(x)$, and $\frac{\partial f}{\partial n^+}(x) = 0$ for each x on ∂E, $\sum_E \phi(x)\Delta f(x) = 0$ which implies that $\sum_T \phi(x)\Delta f(x) = 0$. $\square$

Consequence 4 (Local representation of harmonic functions).

1. *Let E be a finite set of vertices in T and let u be defined on E, harmonic on $\mathring{E}$. Let $a \in \mathring{E}$ and $g_a(x)$ be a superharmonic function on T (Theorem 1) such that $\Delta g_a(x) = -\delta_a(x)$. In Theorem 4(i), take $v(x) = g_a(x)$. Then we have*

$$\phi(a)u(a) = \sum_{\partial E} \phi(s)\left(u\frac{\partial g_a}{\partial n^-} - g_a\frac{\partial u}{\partial n^-}\right).$$

   This result can be interpreted as showing that a harmonic function can be expressed locally by means of a single layer and a double layer potentials on a boundary. (See Brelot [2, p. 179] and Kellogg [5] for the corresponding basic result in the Euclidean case $\mathbf{R}^n$, $n \ge 2$.)

2. In the above representation, instead of choosing E arbitrarily, let $E = \{x : |x| \le m\}$ where the distance $|x| = d(x,e)$ is measured from a fixed vertex e. Let a be some vertex in $\mathring{E}$. Let $h(x)$ be the harmonic function on $\mathring{E}$, such that $h(x) = g_a(x)$ when $x \in \partial E$. (For the existence of this Dirichlet solution h, see Theorem 2.)

   Write $G_a^E(x) = g_a(x) - h(x)$. Then $G_a^E(x)$ is well defined if $|x| \le m$; $G_a^E(x) = 0$ on ∂E; $\Delta G_a^E(x) = -\delta_a(x)$; and the greatest harmonic minorant of $G_a^E(x)$ in $\mathring{E}$ is 0. We can term $G_a^E(x)$ as the Green's function in the ball $|x| \le m$ with pole at the vertex a.

   Now replace the function g_a in the above representation (1) by the Green's function $G_a^E(x)$ to conclude

$$\phi(a)u(a) = \sum_{x \in \partial E} \phi(x)u(x)\frac{\partial G_a^E}{\partial n^-}(x).$$

   This equality expresses the mean-value property for harmonic functions. As an illustration, consider the case of a homogeneous tree T of order $q+1$ ($q \ge 2$ integer); that is, each vertex T has exactly $(q+1)$ neighbours and $p(x,y) = \frac{1}{q+1}$ if $x \sim y$. Let e be a fixed vertex and measure distances from e. Notice in this case $\phi(x) = 1$ for every vertex x. Let $E = \{x : |x| \le m\}$. In this case, $g_e(x) = \frac{q^{1-|x|}}{q-1}$ (Cartier [3, P. 264]) so that $G_e^E(x) = \frac{q^{1-|x|} - q^{1-m}}{q-1}$ and $\frac{\partial G_e^E}{\partial n^-}(x) = \frac{q^{-m+1}}{q+1}$ when $|x| = m$. Hence,

$$u(e) = \frac{1}{(q+1)q^{m-1}} \sum_{|x|=m} u(x).$$

Note that on $|x| = m$, there are $(q+1)q^{m-1}$ vertices.

Conversely, let E be a finite set such that $\mathring{E}$ is connected and a vertex in ∂E has a neighbour in $\mathring{E}$. For any $x \in \mathring{E}$, let $G_x^E(y)$ denote the Green's function on E. Then, given a finite-valued function f on ∂E, $u(x) = \sum_{z \in \partial E} \frac{\phi(z)}{\phi(x)} f(z) \frac{\partial G_x^E}{\partial n^-}(z)$ represents the unique function on E, which is harmonic on $\mathring{E}$ and takes the value f on the boundary ∂E.

For, under the conditions given on E, there exists a unique harmonic function $u(x)$ on $\mathring{E}$ such that $u = f$ on ∂E. Then, as shown above for the case of a ball, we can prove $\phi(x)u(x) = \sum_{z \in \partial E} \phi(z)u(z) \frac{\partial G_x^E}{\partial n^-}(z)$. Since $u(z) = f(z)$ on ∂E, we can write

$$u(x) = \sum_{z \in \partial E} \frac{\phi(z)}{\phi(x)} f(z) \frac{\partial G_x^E}{\partial n^-}(z).$$

Consequence 5 (flux at infinity). Let $E_m = \{x : |x| \le m\}$. Suppose u is a superharmonic function defined on T. Then from Consequence 1 (i), if we take $E = E_m$, we obtain

$$-\sum_{\partial E_m} \phi(s) \frac{\partial u}{\partial n^-}(s) = \sum_{\mathring{E}_m} \phi(x) \Delta u(x).$$

This can be written as

$$-\sum_{|s|=m} \phi(s) \frac{\partial u}{\partial n^-}(s) = \sum_{|x|<m} \phi(x) \Delta u(x),$$

when we realize that $\frac{\partial u}{\partial n^-}(y) = \Delta u(y)$ for any terminal vertex y in E_m. Let us define

$$\operatorname{Flux}(u; E_m) = -\sum_{|s|=m} \phi(s) \frac{\partial u}{\partial n^-}(s) = \sum_{|x|<m} \phi(x) \Delta u(x).$$

Since $\Delta u(x) \le 0$, $\operatorname{Flux}(u; E_m)$ is a decreasing sequence in m. Define the flux at infinity of u as

$$\operatorname{Flux}_\infty u = \lim_{m \to \infty} \operatorname{Flux}(u; E_m).$$

Proposition 5. *If u is a superharmonic function such that $\Delta u = 0$ outside a finite set, then* $\operatorname{Flux}_\infty u$ *is finite.*

Proof. Suppose $u(x)$ is harmonic for $|x| > p$. For a large integer $m > p$, note by Theorem 4 (i),

$$-\left[\sum_{|s|=p} \phi(s) \frac{\partial u}{\partial n^+}(s) + \sum_{|s|=m} \phi(s) \frac{\partial u}{\partial n^-}(s)\right] = \sum_{p<|x|<m} \phi(x) \Delta u(x) = 0.$$

Hence $\mathrm{Flux}(u; E_m) =$ a constant α, for all large $m > p$. This implies $\mathrm{Flux}_\infty\, u = \alpha$. $\square$

Proposition 6. *Let u be a superharmonic function defined outside a finite set in T. Then there exist two superharmonic functions u_1 and u_2 on T, u_2 being harmonic outside a finite set, such that $u = u_1 - u_2$ outside a finite set.*

Proof. Suppose $u(x)$ is defined on T and superharmonic for $|x| \geq m$. Modify $u(x)$ by taking the Dirichlet solution on $\mathring{E}_m$ with boundary values $u(x)$ on E_m. Denote thus extended function also by u. Let $v(x) = u(x) - \sum_{|s|=m} \Delta u(s) g_s(x)$, $g_s(x)$ as in Theorem 1. Then, at each nonterminal vertex x, $\Delta v(x) = \Delta u(x) \leq 0$ if $|x| > m$ and $\Delta v(x) = 0$ if $|x| \leq m$. Hence $v(x)$ is superharmonic on T. We complete the proof of the proposition by remarking $u(x) = \left[v(x) + \sum_{|s|=m} \Delta u(s)^+ g_s(x)\right] - \left[\sum_{|s|=m} \Delta u(s)^- g_s(x)\right]$, when $|x| > m$. $\square$

We can now use the above two propositions to define without ambiguity the flux at infinity of a superharmonic function defined outside a finite set in T.

Suppose u is a superharmonic function defined outside a finite set. Then there exist (Proposition 6) two superharmonic functions u_1 and u_2 defined on T, with u_2 having finite harmonic support, such that $u = u_1 - u_2$ outside a finite set. Note $\mathrm{Flux}_\infty\, u_2$ is finite (Proposition 5). Define $\mathrm{Flux}_\infty\, u = \mathrm{Flux}_\infty\, u_1 - \mathrm{Flux}_\infty\, u_2$.

Note that there is no ambiguity in this definition; for, if $u = v_1 - v_2$ is another such decomposition, then $\sum_{\partial E_m} \phi(s) \frac{\partial(u_1+v_2)}{\partial n^-}(s) = \sum_{\partial E_m} \phi(s) \frac{\partial(v_1+u_2)}{\partial n^-}(s)$, if m is large. Hence

$$\begin{aligned} \mathrm{Flux}_\infty\, u_1 + \mathrm{Flux}_\infty\, v_2 &= \mathrm{Flux}_\infty(u_1 + v_2) \\ &= \mathrm{Flux}_\infty(u_2 + v_1) \\ &= \mathrm{Flux}_\infty\, u_2 + \mathrm{Flux}_\infty\, v_1. \end{aligned}$$

Since $\mathrm{Flux}_\infty\, u_2$ and $\mathrm{Flux}_\infty\, v_2$ are finite, we have

$$\mathrm{Flux}_\infty\, u_1 - \mathrm{Flux}_\infty\, u_2 = \mathrm{Flux}_\infty\, v_1 - \mathrm{Flux}_\infty\, v_2.$$

Lemma 7. *Let u_i be a sequence of superharmonic functions on T, tending to u at each vertex in T. Then u is superharmonic on T, such that* $\mathrm{Flux}_\infty u = \lim_i \mathrm{Flux}_\infty u_i$.

Proof. Since $u(x) = \lim u_i(x)$, at each vertex x we have $\Delta u_i(x) = \sum_y p(x,y)[u_i(y) - u_i(x)] \to \sum_y p(x,y)[u(y) - u(x)] = \Delta u(x)$. Since $\Delta u_i \le 0$, at each nonterminal vertex, we have $\Delta u \le 0$, that is u is superharmonic on T. Now $\mathrm{Flux}(u_i; E_m) = -\sum_{|s|=m} \phi(s) \frac{\partial u_i}{\partial n^-}(s) = \sum_{|x|<m} \phi(x) \Delta u_i(x) = \alpha_{im} \le 0$. Hence

$$\begin{aligned}\mathrm{Flux}_\infty u &= \lim_m \mathrm{Flux}(u; E_m) \\ &= \lim_m \lim_i \alpha_{im} \\ &= \lim_i \lim_m \alpha_{im} \\ &= \lim_i \mathrm{Flux}_\infty u_i. \quad \square\end{aligned}$$

Theorem 8. *Let u be a superharmonic function defined outside a finite set in T. Suppose u has a harmonic minorant outside a finite set. Then* $\mathrm{Flux}_\infty u$ *is finite.*

Proof. We can write (Proposition 6) $u = v - t$ outside a finite set, where v and t are superharmonic on T, t having finite harmonic support. By the assumption, v has a harmonic minorant h in $|x| \ge N$, for some N. It is enough to prove that $\mathrm{Flux}_\infty v$ is finite, thanks to Proposition 5.

Let h_m be the Dirichlet solution in $E = \{x : N \le |x| \le m\}$ with boundary values v. Define

$$v_m = \begin{cases} h_m & \text{on } \mathring{E} \\ v & \text{on } T \backslash \mathring{E}. \end{cases}$$

Then v_m is superharmonic on T, such that $\mathrm{Flux}_\infty v_m = \mathrm{Flux}_\infty v$. Note v_m is decreasing, $v_m \ge h$ when $|x| \ge N$ and $v_m = v$ when $|x| \le N$. Hence $s = \lim_m v_m$ is a superharmonic function on T and $\mathrm{Flux}_\infty s = \lim_m \mathrm{Flux}_\infty v_m$ (Lemma 7). Hence $\mathrm{Flux}_\infty v = \mathrm{Flux}_\infty s < \infty$, since s is harmonic on $|x| > N$ (Proposition 5). $\square$

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