Hiroshima Math. J.
33 (2003), 189–195

Topological entropy and periodic orbits of saddle type for surface diffeomorphisms

Yong Moo Chung and Michihiro Hirayama

(Received March 20, 2002)
(Revised March 4, 2003)

Abstract. It is proved that the topological entropy of a surface diffeomorphism is given by the growth rate of the number of periodic points of saddle type. It is also shown that the number of periodic points with weak hyperbolicity is small.

1. Introduction

It is known by Bowen [1] that if a diffeomorphism $f : M \to M$ of a compact Riemannian manifold M satisfies Axiom A, then the topological entropy $h(f)$ of f holds the following:

$$h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \#Fix(f^n)$$

where $Fix(g)$ denotes the set of fixed points of a map g and $\#A$ the cardinal number of a set A. For a surface diffeomorphism Katok [5] proved that the topological entropy does not exceed the exponential growth rate of the number of periodic points if the derivative of the map is Hölder continuous. However, the formula above is not valid for a diffeomorphism, in general. In fact, Kaloshin [4] showed that the growth rate of the number of periodic points is superexponential for a generic diffeomorphism in a Newhouse domain of the space of C^r diffeomorphisms $(r \geq 2)$.

In this paper, combining some known results in smooth ergodic theory with that obtained by Bowen, we show that the topological entropy of a surface diffeomorphism coincides with the exponential growth rate of the number of periodic points with strong hyperbolicity. Thus none of the numbers of sinks, sources and non-hyperbolic peridic points has influence on the topological entropy. Moreover, we also give an estimate of the number of periodic points with weak hyperbolicity.

2000 *Mathematics Subject Classification*. 37B40, 37C25, 37D05, 37E30.
Key words and phrases. topological entropy, periodic points, Lyapunov exponents, diffeomorphisms of surfaces.

2. Results

Let $f : M \to M$ be a surface diffeomorphism and $Df : TM \to TM$ a derivative of f. Let us fix a smooth Riemannian metric on M. For a point $x \in M$ the numbers

$$\lambda^+(f,x) = \limsup_{n\to+\infty} \frac{1}{n} \log\|Df^n(x)\|,$$

$$\lambda^-(f,x) = \limsup_{n\to-\infty} \frac{1}{n} \log\|Df^n(x)\|$$

are called the *Lyapunov exponents* along the orbit of x, or simply the Lyapunov exponents at x, where we write $Df^n(x) = Df(f^{n-1}x) \circ \cdots \circ Df(x)$ by the chain rule.

Remark. In general, the Lyapunov exponents should be defined by using a decomposition of the tangent space at x. See Chapter 10 of [8] or Supplement of [6] for more details. Since there exist only two exponents at each point x in the case of surface diffeomorphisms, however, we can give it in brief as above.

For a periodic point p of f with period n there is a number γ with $0 < \gamma \le 1$ such that

$$\gamma e^{j\lambda^+(f,p)} \le \|Df^j(f^i(p))\| \le \gamma^{-1}e^{j\lambda^+(f,p)},$$

$$\gamma e^{-j\lambda^-(f,p)} \le \|Df^{-j}(f^i(p))\| \le \gamma^{-1}e^{-j\lambda^-(f,p)}$$

hold for all integers $j \ge 0$ and $0 \le i \le n-1$. A periodic point p is of *saddle* type if $\lambda^-(f,p) < 0 < \lambda^+(f,p)$ holds. For numbers $\alpha, \gamma > 0$ and an integer $n \ge 1$ we set

$$HP_n(f,\alpha,\gamma) = \{p \in Fix(f^n) : \|Df^{\pm j}(f^i(p))\| \ge \gamma e^{j\alpha} \text{ for all } j \ge 0 \text{ and } 0 \le i \le n-1\}.$$

Then $HP_n(f,\alpha,\gamma) \subset HP_n(f,\alpha',\gamma')$ holds if $\alpha \ge \alpha'$, $\gamma \ge \gamma'$, and the set of periodic points of saddle type is given by

$$\bigcup_{\alpha>0} \bigcup_{\gamma>0} \bigcup_{n=1}^{\infty} HP_n(f,\alpha,\gamma).$$

Notice that the invariant subspaces associated with the Lyapunov exponents vary continuously on $\bigcup_{n=1}^{\infty} HP_n(f,\alpha,\gamma)$ for any $\alpha,\gamma > 0$, and that so do the local stable and unstable manifolds. If a periodic point contained in $\bigcup_{n=1}^{\infty} HP_n(f,\alpha,\gamma)$ is close to one of those accumulation points, then it has a transversal homoclinic point for f by the inclination lemma. Applying the Smale homoclinic theorem, furthermore, will deduce $h(f) > 0$. For details, see Chapter 6 of [6]. Thus $h(f) = 0$ implies that $\bigcup_{n=1}^{\infty} HP_n(f, \alpha, \gamma)$ is a finite set for all $\alpha, \gamma > 0$. For a diffeomorphism with positive entropy we shall show the following:

Theorem 1. *For any surface diffeomorphism* $f : M \to M$ *with Hölder continuous derivative*

$$h(f) = \lim_{\gamma \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log \# HP_n(f, \alpha, \gamma)$$

holds whenever $0 < \alpha < h(f)$.

In particular, the topological entropy is characterized by the number of periodic points of saddle type as follows:

Corollary.

$$h(f) = \lim_{\alpha \to 0+} \lim_{\gamma \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log^+ \# HP_n(f, \alpha, \gamma),$$

where $\log^+ a = \max(0, \log a)$.

The formula stated in Theorem 1 is independent of the choice of α. Then it is natural to ask if the exponential growth rate of the number of periodic points with weak hyperbolicity is strictly smaller than the topological entropy. For a number $\beta > 0$ we define a subset of $HP_n(f, \alpha, \gamma)$ by

$$HP_n(f, \alpha, \beta, \gamma) = \{p \in Fix(f^n) : \gamma e^{j\alpha} \le \|Df^{\pm j}(f^i(p))\| \le \gamma^{-1} e^{j\beta}$$
$$\text{for all } j \ge 0 \text{ and } 0 \le i \le n - 1\}.$$

Then $\alpha \le \lambda^+(f, p), -\lambda^-(f, p) \le \beta$ holds for $p \in HP_n(f, \alpha, \beta, \gamma)$, and it is obvious that

$$HP_n(f, \alpha, \gamma) = \bigcup_{\beta > 0} HP_n(f, \alpha, \beta, \gamma) = HP_n(f, \alpha, \beta_0, \gamma)$$

if $0 < \gamma \le 1$, where $\beta_0 = \max\{\log\|Df(x)\|, \log\|Df^{-1}(x)\| : x \in M\}$. Another result of this paper is the following:

Theorem 2. *For any surface diffeomorphism* $f : M \to M$ *and numbers* $\alpha, \beta, \gamma > 0$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \# HP_n(f, \alpha, \beta, \gamma) \le \beta$$

holds.

REMARK. The Hölder continuity of the derivative is not assumed in Theorem 2.

Some results corresponding to Theorems 1 and 2 for the expanding periodic orbits of one-dimensional maps can be found in [2, 3].

3. Proofs

We need the notion of hyperbolic set to prove the theorems. A compact f-invariant set Λ is called *hyperbolic* for f if there exists a Df-invariant splitting $T_\Lambda M = E^u \oplus E^s$ with constants $\lambda, \gamma > 0$ such that

$$\|Df^j(x)v\| \geq \gamma e^{j\lambda}\|v\| \qquad (v \in E_x^u),$$

$$\|Df^j(x)w\| \leq \gamma^{-1} e^{-j\lambda}\|w\| \qquad (w \in E_x^s)$$

for all $x \in \Lambda$ and integers $j \geq 0$. In addition, Λ is said to be *isolated* if

$$\bigcap_{n=-\infty}^{\infty} f^{-n} U = \Lambda$$

holds for some neighborhood U of Λ. A hyperbolic set Λ of f is isolated if and only if f has a local product structure on Λ. See Theorem 18.4.1 and Proposition 6.4.21 of [6]. The following proposition obtained by Bowen is important in our proof.

PROPOSITION ([1], Theorem 18.5.1 of [6]). *For a hyperbolic set Λ of f,*

$$\limsup_{n \to \infty} \frac{1}{n} \log \#Fix(f^n|_\Lambda) \leq h(f|_\Lambda)$$

holds. Moreover, the equality holds if Λ is isolated.

PROOF OF THEOREM 2. For any numbers $\alpha, \beta, \gamma > 0$ we show the following:

$$\limsup_{n \to \infty} \frac{1}{n} \log \#HP_n(f, \alpha, \beta, \gamma) \leq \min\{\beta, h(f)\} \tag{1}$$

that involves Theorem 2. Let

$$\Gamma = \Gamma_{\alpha,\beta,\gamma} = \mathrm{cl}\big(\textstyle\bigcup_{n=1}^{\infty} HP_n(f, \alpha, \beta, \gamma)\big)$$

where $\mathrm{cl}(A)$ denotes the closure of a set A. The invariant splitting $T_x M = E_x^u \oplus E_x^s$ associated with the Lyapunov exponents varies continuously on

$\bigcup_{n=1}^{\infty} HP_n(f,\alpha,\beta,\gamma)$ and hence it can be extended to Γ. Therefore Γ is either an empty set or a hyperbolic set of f. In the former case, $HP_n(f,\alpha,\beta,\gamma)$ is also empty for all integers $n \geq 1$, and hence (1) holds. Otherwise, since

$$Fix(f^n|_\Gamma) = HP_n(f,\alpha,\beta,\gamma)$$

for any $n \geq 1$, by the proposition we have

$$\begin{aligned}(2)\qquad \limsup_{n\to\infty} \frac{1}{n} \log \#HP_n(f,\alpha,\beta,\gamma) &= \limsup_{n\to\infty} \frac{1}{n} \log \#Fix(f^n|_\Gamma) \\ &\leq h(f|_\Gamma) \\ &\leq h(f).\end{aligned}$$

Then the variational principle for topological entropy, see e.g. Corollary 8.6.1 of [8], asserts that for any number $\varepsilon > 0$ there is an f-invariant ergodic Borel probability measure μ supported on Γ such that

$$(3)\qquad h(f|_\Gamma) \geq h_\mu(f) \geq h(f|_\Gamma) - \varepsilon$$

where $h_\mu(f)$ denotes the metric entropy of μ for f. The ergodicity of the measure μ implies that the Lyapunov exponents at x are constants for almost everywhere and are denoted by $\lambda_\mu^+(f)$ and $\lambda_\mu^-(f)$, respectively. Then by the Ruelle inequality [7] we have

$$h_\mu(f) \leq \min\{\lambda_\mu^+(f), -\lambda_\mu^-(f)\}.$$

Since

$$\alpha \leq \lambda^+(f,x) \leq \beta, \qquad \alpha \leq -\lambda^-(f,x) \leq \beta$$

holds for all $x \in \Gamma$, we have

$$\alpha \leq \lambda_\mu^+(f) \leq \beta, \qquad \alpha \leq -\lambda_\mu^-(f) \leq \beta,$$

and then

$$(4)\qquad h_\mu(f) \leq \beta.$$

Combining (2) with (3), (4), and letting $\varepsilon \to 0$, we obtain (1), and hence Theorem 2. □

Proof of Theorem 1. It follows from (1) that

$$\limsup_{n\to\infty} \frac{1}{n} \log \#HP_n(f,\alpha,\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log \#HP_n(f,\alpha,\beta_0,\gamma) \leq h(f)$$

holds for any numbers $\alpha > 0$ and γ with $0 < \gamma \le 1$. Thus for the proof of Theorem 1 it remains to show that

$$h(f) \le \lim_{\gamma \to 0+} \limsup_{n\to\infty} \frac{1}{n} \log \# HP_n(f, \alpha, \gamma) \tag{5}$$

for any α with $0 < \alpha < h(f)$ under the assumption of the Hölder continuity of the derivative for f. By the variational principle and the Ruelle inequality for any number ε with $0 < 2\varepsilon < h(f) - \alpha$ there is an f-invariant ergodic Borel probability measure μ such that

$$\min\{\lambda_\mu^+(f), -\lambda_\mu^-(f)\} \ge h_\mu(f) \ge h(f) - \varepsilon > \alpha + \varepsilon. \tag{6}$$

Moreover, since f has Hölder continuous derivative, by the Katok theorem [5]; see also Theorem S.5.9 of [6], there exists an isolated hyperbolic set Λ of f with a Df-invariant splitting $T_\Lambda M = E^u \oplus E^s$ and $\gamma_0 > 0$ such that

$$\begin{aligned} h(f|_\Lambda) &\ge h_\mu(f) - \varepsilon, \\ \|Df^j(x)v\| &\ge \gamma_0 e^{j(\lambda_\mu^+(f)-\varepsilon)} \|v\| \qquad (v \in E_x^u), \\ \|Df^j(x)w\| &\le \gamma_0^{-1} e^{j(\lambda_\mu^-(f)+\varepsilon)} \|w\| \qquad (w \in E_x^s) \end{aligned}$$

for all $x \in \Lambda$ and integers $j \ge 0$. It follows from (6) that

$$Fix(f^n|_\Lambda) \subset HP_n(f, \alpha, \gamma_0)$$

for any integer $n \ge 1$, and hence

$$\begin{aligned} h(f) - 2\varepsilon &\le h_\mu(f) - \varepsilon \\ &\le h(f|_\Lambda) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \# Fix(f^n|_\Lambda) \\ &\le \limsup_{n\to\infty} \frac{1}{n} \log \# HP_n(f, \alpha, \gamma_0) \\ &\le \lim_{\gamma\to 0+} \limsup_{n\to\infty} \frac{1}{n} \log \# HP_n(f, \alpha, \gamma) \end{aligned}$$

by the proposition. Letting $\varepsilon \to 0$ we obtain (5). This completes the proof of Theorem 1. $\square$

References

[1] R. Bowen, Topological entropy and Axiom A, Global Analysis (Berkeley, CA, 1968), Proc. Sympos. Pure Math. **14**, Amer. Math. Soc., Providence, RI, 1970, pp. 23–41.

[2] Y. M. Chung, Topological entropy for differentible maps of intervals, Osaka J. Math. **38** (2001), 1–12.

[3] Y. M. Chung, Expanding periodic orbits with small exponents, J. Difference Equ. Appl. **9** (2003), 337–341.

[4] V. Y. Kaloshin, An extension of the Artin-Mazur theorem, Ann. of Math. **150** (1999), 729–741.

[5] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Inst. Hautes Etudes Sci. Publ. Math. **51** (1980), 137–173.

[6] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Encyclopedia of Mathematics and its Applications **54**, Cambridge Univ. Press, Cambridge, 1995.

[7] D. Ruelle, An inequality for the entropy of differentiable maps, Bol. Soc. Brasil. Math. **9** (1978), 83–87.

[8] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics **79**, Springer-Verlag, New York, 1982.

Yong Moo Chung
Department of Applied Mathematics
Graduate School of Engineering
Hiroshima University
Higashi-Hiroshima 739-8527*, Japan*
E-mail address: chung@amath.hiroshima-u.ac.jp

Michihiro Hirayama
Graduate School of Mathematics
Kyushu University
Fukuoka 812-8581*, Japan*
E-mail address: hirayama@math.kyushu-u.ac.jp