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Comparison theorems of two Adams spectral sequences

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ABSTRACT. Suppose that $f : X \to Y$ is a map between spectra and $E_2(X), F_2(Y)$ are E_2-terms of the E-, F-Adams spectral sequences for X, Y, respectively. We shall give conditions for $f_* : E_2(X) \to F_2(Y)$ such that $f_* : \pi_t(X) \to \pi_t(Y)$ is monomorphic or epimorphic.

1. Introduction

Let $f : X \to Y$ be a map of spectra. In this paper, we argue conditions such that the homomorphism $f_* : \pi_t(X) \to \pi_t(Y)$ is monomorphic or epimorphic by using the Adams spectral sequences. Let $\lambda : E \to F$ be a map of ring spectra. Then we have the E- and F-Adams spectral sequences $\{E_r^{s,t}(X)\}$ and $\{F_r^{s,t}(Y)\}$ abutting to $\pi_{t-s}(X)$ and $\pi_{t-s}(Y)$, respectively, and a homomorphism $f_* \circ \lambda_* : \{E_r^{s,t}(X)\} \to \{F_r^{s,t}(Y)\}$. We denote

$$\bar{Z}E_r^{s,t}(X) = \{x \in E_r^{s,t}(X) \mid x \text{ converges to some element of } \pi_*(X)\},$$

$$\bar{Z}F_r^{s,t}(Y) = \{y \in F_r^{s,t}(Y) \mid y \text{ converges to some element of } \pi_*(Y)\}.$$

Our main theorems are the following.

THEOREM 1.1. *Suppose that $\{E_r^{s,t+s}(X)\}$ converges to $\pi_t(X)$. Fix an integer t. We assume the following:*
i) *There exists an integer $s_0(t)$ such that $E_\infty^{s,t+s}(X) = 0$ for $s > s_0(t)$.*
ii) *$f_* \circ \lambda_* : \bar{Z}E_2^{s,t+s}(X) \to \bar{Z}F_2^{s,t+s}(Y)$ is monomorphic for $0 \leq s \leq s_0(t)$.*
iii) *$f_* \circ \lambda_* : E_2^{s,t+s+1}(X) \to F_2^{s,t+s+1}(Y)$ is epimorphic for $0 \leq s \leq s_0(t) - 2$.*
Then $f_* : \pi_t(X) \to \pi_t(Y)$ is monomorphic.

THEOREM 1.2. *Suppose that $\{F_r^{s,t+s}(Y)\}$ converges to $\pi_t(Y)$. Fix an integer t. We assume the following:*
i) *There exists an integer $s_1(t)$ such that $F_\infty^{s,t+s}(Y) = 0$ for any $s > s_1(t)$*
ii) *$f_* \circ \lambda_* : \bar{Z}E_2^{s,t+s}(X) \to \bar{Z}F_2^{s,t+s}(Y)$ is epimorphic for $0 \leq s \leq s_1(t)$.*
Then $f_* : \pi_t(X) \to \pi_t(Y)$ is epimorphic.

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If E_*E is flat, then $E_2^{s,t}(X) = \mathrm{Ext}_{E_*E}^{s,t}(E_*, E_*(X))$. But we argue in the general cases. As an application of the main theorems, consider the localizations $L_E X$ and $L_F X$. The ring map λ induces a map $L_\lambda : L_E X \to L_F X$. We apply the main theorems to L_λ. We define a spectrum $\overline{E}$ by a cofibration $S^0 \xrightarrow{\eta^E} E \to \overline{E}$ for a unit η^E of E and $\overline{E}^n = \overline{E} \wedge \cdots \wedge \overline{E}$ (n-times). Consider the edge homomorphism $\phi^F : \pi_t(X) \to F_2^{0,t}(X)$. This is induced from the Hurewicz homomorphism $\pi_t(X) \to F_t(X)$.

Theorem 1.3. *Suppose that* $\{E_r^{s,t+s}(X)\}$ *converges to* $\pi_t(L_E X)$. *Fix an integer* t. *We assume the following:*

i) *There exists an integer* $s_0(t)$ *such that* $E_\infty^{s,t+s}(X) = 0$ *for* $s > s_0(t)$.

ii) $F_2^{s,t+s+r+1}(E \wedge \overline{E}^r \wedge X) = 0$ *for* $0 < s < s_0(t) - r$.

iii) $\phi^F : \pi_{u+s}(E \wedge \overline{E}^s \wedge X) \to F_2^{0,u+s}(E \wedge \overline{E}^s \wedge X)$ *is monomorphic for* $s \le s_0(t)$ *and* $u = t$, *and epimorphic for* $s < s_0(t)$ *and* $u = t+1$.

Then $L_{\lambda *} : \pi_t(L_E X) \to \pi_t(L_F X)$ *is monomorphic.*

Theorem 1.4. *Suppose that* $\{E_r^{s,t+s}(X)\}$ *and* $\{F_r^{s,t+s}(X)\}$ *converge to* $\pi_t(L_E X)$ *and* $\pi_t(L_F X)$, *respectively. Fix an integer* t. *We assume the following:*

i) *There exists an integer* $s_1(t)$ *such that* $F_\infty^{s,t+s}(X) = 0 = \overline{Z}E_2^{s+1,t+s}(X)$ *for any* $s > s_1(t)$.

ii) $F_2^{s,t+s+r}(E \wedge \overline{E}^r \wedge X) = 0$ *for* $0 < s < s_1(t) - r + 1$.

iii) $\phi^F : \pi_{u+s}(E \wedge \overline{E}^s \wedge X) \to F_2^{0,u+s}(E \wedge \overline{E}^s \wedge X)$ *is monomorphic for* $s \le s_1(t) + 1$ *and* $u = t-1$, *and epimorphic for* $s < s_1(t) + 1$ *and* $u = t$.

Then $L_{\lambda *} : \pi_t(L_E X) \to \pi_t(L_F X)$ *is epimorphic.*

Corollary 1.5. *Suppose that* $E_*(E)$ *and* $E_*(X)$ *are flat over* $E_*(S^0)$, $F_*(F)$ *is flat over* $F_*(S^0)$, $\mathrm{Ext}_{F_*F}^{s,*}(F_*, F_*(E)) = 0$ *for* $0 < s < s_0$ *and* $\phi^F : \pi_*(E) \to \mathrm{Ext}_{F_*F}^{0,*}(F_*, F_*(E))$ *is isomorphic. Moreover we assume that* $\{E_r^{s,t+s}(X)\}$ *and* $\{F_r^{s,t+s}(X)\}$ *converge to* $\pi_t(L_E X)$ *and* $\pi_t(L_F X)$, *respectively.*

Fix an integer t. *If* $\mathrm{Ext}_{E_*E}^{s,t+s}(E_*, E_*(X)) = 0$ *for* $s > s_0$, *then* $L_{\lambda *} : \pi_t(L_E X) \to \pi_t(L_F X)$ *is monomorphic. If* $\mathrm{Ext}_{F_*F}^{s,t+s}(F_*, F_*(X)) = 0 = \mathrm{Ext}_{E_*E}^{s+1,t+s}(E_*, E_*(X))$ *for* $s > s_0 - 1$ *then* $L_{\lambda *} : \pi_t(L_E X) \to \pi_t(L_F X)$ *is epimorphic.*

For our purpose, we review the theory of the Adams spectral sequence in §2 and compare two Adams spectral sequences in §3. Theorems 1.1–1.4 and Corollary 1.5 are proved at §3. In this paper, we work in the stable homotopy category.

2. The Adams spectral sequences

Let E be a ring spectrum with unit $\eta^E : S^0 \to E$. Consider a cofibering $S^0 \xrightarrow{\eta^E} E \xrightarrow{\bar{\eta}^E} \overline{E}$ and the boundary homomorphism $\partial^E : \pi_{t+1}(\overline{E} \wedge X) \to \pi_t(S^0 \wedge X)$. For a spectrum X, we denote

$$X_s^E = \bar{E}^s \wedge X \qquad (\bar{E}^s = \bar{E} \wedge \cdots \wedge \bar{E}) \qquad \text{and cofibrations}$$

$$S^0 \wedge X_s^E \xrightarrow{\eta^E} E \wedge X_s^E \xrightarrow{\bar{\eta}^E} X_{s+1}^E \tag{2.1}$$

The boundary homomorphisms $\partial^E : \pi_{t+1}(X_{s+1}^E) \to \pi_t(X_s^E)$ define the Adams filtration

$$\pi_{t+s}(X_s^E) \to \pi_{t+s-1}(X_{s-1}^E) \to \cdots \to \pi_t(X_0^E) = \pi_t(X). \tag{2.2}$$

Then we have the E-Adams spectral sequence $\{E_r^{s,t}(X), d_r^E : E_r^{s,t}(X) \to E_r^{s+r,t+r-1}(X)\}$. The E_1 and E_2 terms are as follows:

$$E_1^{s,t}(X) = \pi_t(E \wedge X_s^E) = E_t(X_s^E), \quad d_1^E : \pi_t(E \wedge X_s^E) \xrightarrow{\bar{\eta}_*^E} \pi_t(X_{s+1}^E) \xrightarrow{\eta_*^E} \pi_t(E \wedge X_{s+1}^E),$$

$$E_2^{s,t}(X) = \text{Ker } d_1^E/\text{Im } d_1^E.$$

By definition, $\text{Im}[\eta_*^E : \pi_t(X_s^E) \to \pi_t(E \wedge X_s^E)] \subset \text{Ker } d_1^E$ and $\text{Ker}[\bar{\eta}_*^E : \pi_t(E \wedge X_s^E) \to \pi_t(X_{s+1}^E)] \supset \text{Im } d_1^E$, and so we have homomorphisms

$$\eta_*^E : \pi_t(X_s^E) \to E_2^{s,t}(X), \qquad \bar{\eta}_*^E : E_2^{s,t}(X) \to \pi_t(X_{s+1}^E). \tag{2.3}$$

Remark 2.1. If $E_*E = E_*(E)$ is flat over $E_* = E_*(S^0)$, then $E_2^{s,t}(X) = \text{Ext}_{E_*E}^{s,t}(E_*, E_*(X))$ and the homomorphism $\eta_*^E : \pi_t(X_s^E) \to E_2^{s,t}(X)$ is the composition of the Hurewicz homomorphism $\phi^E : \pi_t(X_s^E) \to \text{Hom}_{E_*E}^t(E_*, E_*(X_s^E)$ and the coboundary homomorphisms

$$\text{Hom}_{E_*E}^t(E_*, E_*(X_s^E)) \to \text{Ext}_{E_*E}^1(E_*, E_*(X_{s-1}^E)) \cong \cdots \cong \text{Ext}_{E_*E}^{s,t}(E_*, E_*(X)).$$

We use notation

$$ZE_2^{s,t}(X) = \{x \in E_2^{s,t}(X) \mid d_r^E x = 0 \text{ in } E_r\text{-term for any } r\} \qquad \text{and}$$

$$\bar{Z}E_2^{s,t}(X) = \text{Im}[\eta_*^E : \pi_t(X_s^E) \to E_2^{s,t}(X)] \subset ZE_2^{s,t}(X).$$

Remark 2.2. We notice that $\bar{Z}E_2^{s,t}(X)$ depends on the Adams resolution (2.2). Let $f : X \to Y$ be a map such that $f_* : E_2^{s,t}(X) \to E_2^{s,t}(Y)$ is isomorphic. Then $f_* : ZE_2^{s,t}(X) \to ZE_2^{s,t}(Y)$ is isomorphic, but $f_* : \bar{Z}E_2^{s,t}(X) \to \bar{Z}E_2^{s,t}(Y)$ is not so. For example, consider the E-localization map $X \to L_E X$. If X is E-prenilpotent then $\{E_r^{s,t}(X)\}$ converges to $\pi_*(L_E X)$ and, by [2],

$$E_2^{s,t}(X) = E_2^{s,t}(L_E X) \qquad \text{and}$$

$$ZE_2^{s,t}(X) = ZE_2^{s,t}(L_E X) = \bar{Z}E_2^{s,t}(L_E X) \supset \bar{Z}E_2^{s,t}(X). \tag{2.4}$$

The following lemma holds by definition (see [4]).

Lemma 2.3. i) *$x_s \in \pi_t(X_s^E)$ satisfies $\eta_*^E(x_s) = 0 \in E_2^{s,t}(X)$ if and only if there are elements $x_{s+1} \in \pi_{t+1}(X_{s+1}^E)$ and $w_{s-1} \in \pi_t(E \wedge X_{s-1}^E)$ with $x_s = \partial^E(x_{s+1}) + \bar{\eta}_*(w_{s-1})$.*

ii) *$w \in \pi_t(E \wedge X_s^E)$ satisfies $d_1^E w = 0$ if and only if there exists an element $x_{s+2} \in \pi_{t+1}(X_{s+2}^E)$ with $\bar{\eta}_*^E(w) = \partial^E(x_{s+2}) \in \pi_t(X_{s+1}^E)$.*

The following is a well-known fundamental result.

PROPOSITION 2.4. i) *$x^E \in E_2^{s,t}(X)$ converges to $x \in \pi_{t-s}(X)$ if and only if there exists an element $x_s \in \pi_t(X_s^E)$ such that $x^E = \eta_*^E(x_s)$ and $(\partial^E)^s(x_s) = x$.*

ii) *$y^E = d_r^E(x^E)$ in E_r-term for $x^E \in E_2^{s,t}(X)$ and $y^E \in E_2^{s+r,t+r-1}(X)$ if and only if there exists an element $y_{s+r} \in \pi_{t+r-1}(X_{s+r}^E)$ such that $\bar{\eta}_*^E(x^E) = (\partial^E)^{r-1}(y_{s+r})$ and $\eta_*^E(y_{s+r}) = y^E$.*

iii) *$\bar{Z}E_2^{s,t}(X) \subset ZE_2^{s,t}(X)$. Especially, $\bar{Z}E_2^{s,t}(X) = ZE_2^{s,t}(X)$ if and only if $\{E_r^{s,t}(X)\}$ converges to $\pi_{t-s}(X)$.*

iv) *If $\{E_r^{s,t}(X)\}$ converges to $\pi_{t-s}(X)$ and there exists an integer $s(t)$ such that $E_\infty^{s,s+t}(X) = 0$ for $s > s(t)$, then*

$$\mathrm{Im}\{(\partial^E)^s : \pi_{t+s}(X_s^E) \to \pi_t(X)\} = 0 \qquad \textit{for } s > s(t).$$

DEFINITION 2.5. Let $\{d_r^E x^E\} \subset E_2^{s+r,t+r-1}(X)$ be a subset consisting of elements y^E satisfying the above proposition ii) for an element $x^E \in E_2^{s,t}(X)$.

COROLLARY 2.6. i) *$\{d_2^E x^E\} = d_2^E(x^E)$ and $\{d_r^E 0\} = \bigcup_{x^E \in E_2^{s+1,t+1}(X)} \{d_{r-1}^E x^E\}$ $(0 \in E_2^{s,t}(X))$. If $d_r^E(x^E) \neq 0$ in E_r-term, then $\{d_{r+1}^E x^E\} = \emptyset$.*

ii) *If $y_1^E, y_2^E \in \{d_r^E x^E\}$ for $x^E \in E_2^{s,t}(X)$, then there exists an element $\theta^E \in E_2^{s+1,t+1}(X)$ such that $y_1^E - y_2^E \in \{d_{r-1}^E \theta^E\}$. Moreover if $y_1^E \neq y_2^E$, then there exists an element $\theta^E \neq 0 \in E_2^{s+a,t+a}(X)$ $(1 \leq a \leq r-2)$ such that $y_1^E - y_2^E \in \{d_{r-a}^E \theta\}$.*

iii) *$\{d_r^E x^E\} \ni 0$ if and only if $\{d_{r+1}^E x^E\} \neq \emptyset$. Hence $x^E \in ZE_2^{s,t}(X)$ if and only if $\{d_r^E x^E\} \ni 0$ for any $r \geq 2$.*

iv) *If $(\partial^E)^{r-1}(x_s) \neq 0$ and $(\partial^E)^r(x_s) = 0$ for $x_s \in \pi_t(X_s^E)$ and $2 \leq r \leq s$, then there exists an element $y^E \neq 0 \in E_2^{s-r,t-r+1}(X)$ such that $\{d_r^E y^E\} \ni \eta_*^E(x_s)$.*

Consider another ring spectrum F with unit $\eta^F : S^0 \to F$ and a ring map $\lambda : E \to F$. We have cofiberings $X_s^F \to F \wedge X_s^F \to X_{s+1}^F$ and the Adams spectral sequence $\{F_r^{s,t}(X)\}$ abutting to $\pi_{t-s}(X)$. Then λ induces maps $\lambda_0 = id : X \to X, \lambda_s : X_s^E \to X_s^F$ inductively by the commutative diagrams:

$$\begin{array}{ccccc} X_s^E & \xrightarrow{\eta^E} & E \wedge X_s^E & \xrightarrow{\bar{\eta}^E} & X_{s+1}^E \\ \downarrow{\scriptstyle \lambda_s} & & \downarrow{\scriptstyle \lambda \wedge \lambda_s} & & \downarrow{\scriptstyle \lambda_{s+1}} \\ X_s^F & \xrightarrow{\eta^F} & F \wedge X_s^F & \xrightarrow{\bar{\eta}^F} & X_{s+1}^F. \end{array} \qquad (2.5)$$

Hence we have a homomorphism $\lambda_* : \{E_r^{s,t}(X)\} \to \{F_r^{s,t}(X)\}$ between spectral sequences.

We notice that $\eta_*^E : F_*(X_s^E) \to F_*(E \wedge X_s^E)$ is monomorphic by the inverse map $F \wedge E \wedge X_s^E \to F \wedge F \wedge X_s^E \to F \wedge X_s^E$. Consider the following conditions for some integers $a \geq 0, b$:

$$F_2^{s,s+r+b+1}(E \wedge X_r^E) = 0 \qquad \text{for } 0 < s < a - r. \tag{2.6}$$

$\phi^F = \eta_*^F : \pi_{t+s}(E \wedge X_s^E) \to F_2^{0,t+s}(E \wedge X_s^E)$

is monomorphic for $s \leq a,\ t = b$ and epimorphic for $s < a,\ t = b+1$.

Then we have [4, Theorem 3.5].

Theorem 2.7. *For integers $a \geq 0$ and b satisfying* (2.6), *the following holds:*

i) $\lambda_* : E_2^{s,t+s}(X) \to F_2^{s,t+s}(X)$ *is monomorphic for $s \leq a$ and $t = b$ and epimorphic for $s < a$ and $t = b+1$.*

ii) $\lambda_* : \bar{Z}E_2^{s,s+b}(X) \to \bar{Z}F_2^{s,s+b}(X)$ *is isomorphic for $s \leq a$ and epimorphic for $s = a+1$.*

3. Comparison of two Adams spectral sequences

In this section, we compare two Adams spectral sequences and prove Theorems 1.1, 1.4 and Corollary 1.5.

Let $f : X \to Y$ be a map between spectra and $\lambda : E \to F$ a ring map between ring spectra. Inductively, we have maps $f_0 = f : X \to Y, f_s : X_s^E \to Y_s^F$ by the commutative diagrams:

$$\begin{array}{ccccccc}
X_s^E & \xrightarrow{\eta^E} & E \wedge X_s^E & \xrightarrow{\bar{\eta}^E} & X_{s+1}^E & \xrightarrow{\partial^E} & \Sigma X_s^E \\
\downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle \lambda \wedge f_s} & & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle \Sigma f_s} \\
Y_s^F & \xrightarrow{\eta^F} & F \wedge Y_s^F & \xrightarrow{\bar{\eta}^F} & Y_{s+1}^F & \xrightarrow{\partial^F} & \Sigma Y_s^F.
\end{array}$$

Then we have a homomorphism $f_* \circ \lambda_* : \{E_2^{s,t}(X)\} \to \{F_2^{s,t}(Y)\}$ between spectral sequences.

Now we prepare the following.

Lemma 3.1. i) *Let $x_s \in \pi_t(X_s^E)$ and $y_{s+1} \in \pi_{t+1}(Y_{s+1}^F)$ be elements with $f_{s-1*} \circ \partial^E(x_s) = (\partial^F)^2 y_{s+1}$. If $f_* \circ \lambda_* : \bar{Z}E_2^{s,t}(X) \to \bar{Z}F_2^{s,t}(Y)$ is monomorphic, then there exists an element $x_{s+1} \in \pi_{t+1}(X_{s+1}^E)$ with $(\partial^E)^2 x_{s+1} = \partial^E x_s$.*

ii) *Let $x_{s+1} \in \pi_{t+1}(X_{s+1}^E)$ and $y_{s+1} \in \pi_{t+1}(Y_{s+1}^F)$ be elements with $f_{s-1*} \circ (\partial^E)^2(x_{s+1}) = (\partial^F)^2 y_{s+1}$. If $f_* \circ \lambda_* : E_2^{s-1,t}(X) \to F_2^{s-1,t}(Y)$ is epimorphic, then there exists an element $x'_{s+1} \in \pi_{t+1}(X_{s+1}^E)$ such that*

$$(\partial^E)^2 x'_{s+1} = (\partial^E)^2 x_{s+1} \qquad \textit{and} \qquad f_{s*} \circ \partial^E(x'_{s+1}) = \partial^F y_{s+1}.$$

Proof. i) By $\partial^F(f_{s*}x_s - \partial^F y_{s+1}) = 0$, we have $v \in \pi_t(F \wedge Y^F_{s-1})$ with

$$f_{s*}x_s = \partial^F y_{s+1} + \bar{\eta}^F_* v. \tag{3.1}$$

Then $f_{s*}(\eta^E_* x_s) = \eta^F_* \circ \bar{\eta}^F_*(v) = d^F_1 v$, and so $f_* \circ \lambda_*(\eta^E_* x_s) = 0 \in \bar{Z}F^{s,t}_2(Y)$. By the assumption, $\eta^E_* x_s = 0 \in \bar{Z}E^{s,t}_2(X)$. By Lemma 2.3 i), we have elements $x_{s+1} \in \pi_{t+1}(X^E_{s+1})$ and $w \in \pi_t(E \wedge X^E_{s-1})$ with $x_s = \partial^E(x_{s+1}) + \bar{\eta}^E_*(w)$. Hence $(\partial^E)^2 x_{s+1} = \partial^E x_s$.

ii) By $(\partial^F)^2(f_{s+1*}x_{s+1} - y_{s+1}) = 0$, we have $v \in \pi_t(F \wedge Y^F_{s-1})$ with

$$f_{s*} \circ \partial^E(x_{s+1}) = \partial^F y_{s+1} + \bar{\eta}^F_* v. \tag{3.2}$$

By $d^F_1 v = \eta^F_* \circ \bar{\eta}^F_* v = 0$ and the assumption, we have an element $w \in E^{s-1,t}_2(X)$ with $f_* \circ \lambda_*(w) = v \in F^{s-1,t}_2(Y)$. Then

$$\eta^E_* \circ \bar{\eta}^E_* w = d^E_1 w = 0 \qquad \text{and} \qquad f_{s*} \circ \bar{\eta}^E_* w = \bar{\eta}^F_* v,$$

and so we have $x' \in \pi_{t+1}(X^E_{s+1})$ with $\partial^E x' = \bar{\eta}^E_* w$. Now we take $x'_{s+1} = x_{s+1} - x'$. Then

$$(\partial^E)^2 x'_{s+1} = (\partial^E)^2 x_{s+1} \qquad \text{and}$$

$$f_{s*} \circ \partial^E(x'_{s+1}) = \partial^F y_{s+1} + \bar{\eta}^F_* v - f_{s*} \circ \bar{\eta}^E_* w = \partial^F y_{s+1}. \qquad \square$$

We use this lemma inductively. By the statements i), ii) applied for $s+i$ instead of s with $0 \le i \le n-2$, we can define elements $x_{s+1}, x_{s+2}, \dots, x_{s+n-1}$ inductively with

$$(\partial^E)^2 x_{s+i+1} = \partial^E x_{s+i} \qquad \text{and} \qquad f_{s+i*} \circ (\partial^E)^2(x_{s+i+1}) = (\partial^F)^{n-i} y_{s+n}$$

and finally by the statement i) for $i = n-1$, we have the desired element x_{s+n} in the following corollaries.

Lemma 3.2. *Let $x_s \in \pi_t(X^E_s)$ and $y_{s+n} \in \pi_{t+n}(Y^F_{s+n})$ be elements with $f_{s-1*} \circ \partial^E(x_s) = (\partial^F)^{n+1} y_{s+n}$. We assume the following:*
i) *$f_* \circ \lambda_* : \bar{Z}E^{s+i,t+i}_2(X) \to \bar{Z}F^{s+i,t+i}_2(Y)$ is monomorphic for $0 \le i \le n-1$.*
ii) *$f_* \circ \lambda_* : E^{s+i-1,t+i}_2(X) \to F^{s+i-1,t+i}_2(Y)$ is epimorphic for $0 \le i \le n-2$.*
Then there exists an element $x_{s+n} \in \pi_{t+n}(X^E_{s+n})$ such that

$$(\partial^E)^{n+1} x_{s+n} = \partial^E x_s \qquad \text{and} \qquad f_{s+n-2*} \circ (\partial^E)^2(x_{s+n}) = (\partial^F)^2 y_{s+n}.$$

Moreover, if $f_* \circ \lambda_* : E^{s+n-2,t+n-1}_2(X) \to F^{s+n-2,t+n-1}_2(Y)$ is epimorphic, then we can take the above x_{s+n} such that $f_{s+n-1*} \circ \partial^E(x_{s+n}) = \partial^F y_{s+n}$.

Corollary 3.3. *Let x^E be an element of $E^{s,t}_2(X)$. We assume the following:*
i) *$f_* \circ \lambda_* : \bar{Z}E^{s+i,t+i-1}_2(X) \to \bar{Z}F^{s+i+1,t+i}_2(Y)$ is monomorphic for $2 \le i \le r-1$.*
ii) *$f_* \circ \lambda_* : E^{s+i,t+i}_2(X) \to F^{s+i,t+i}_2(Y)$ is epimorphic for $1 \le i \le r-3$.*

Under these conditions, if $\{d_r^F(f_* \circ \lambda_*(x^E))\} \neq \emptyset$, *then* $\{d_r^E(x^E)\} \neq \emptyset$.

Moreover, if $f_* \circ \lambda_* : E_2^{s+r-2,t+r-2}(X) \to F_2^{s+r-2,t+r-2}(Y)$ *is epimorphic, then the induced map* $f_* \circ \lambda_* : \{d_r^E x^E\} \to \{d_r^F(f_* \circ \lambda_*(x^E))\}$ *is surjective.*

Proof. For any $y^F \in \{d_r^F(f_* \circ \lambda_*(x^E))\} \subset F_2^{s+r,t+r-1}(Y)$, we have an element $y_{s+r} \in \pi_{t+r-1}(Y_{s+r}^F)$ with

$$\bar{\eta}_*^F(f_* \circ \lambda_*(x^E)) = (\partial^F)^{r-1} y_{s+r} \qquad \text{and} \qquad \eta_*^F y_{s+r} = y^F$$

by Definition 2.5. On the other hand, we have an element $x_{s+2} \in \pi_{t+1}(X_{s+2}^E)$ with $\partial^E x_{s+2} = \bar{\eta}_*^E x^E$ by Lemma 2.3 ii). These imply that

$$f_{s+1*} \circ \partial^E x_{s+2} = f_{s+1*} \circ \bar{\eta}_*^E x^E = \bar{\eta}_*^F(f_* \circ \lambda_*(x^E)) = (\partial^F)^{r-1} y_{s+r}.$$

By Lemma 3.2, we have an element $x_{s+r} \in \pi_{t+r-1}(X_{s+r}^E)$ with

$$(\partial^E)^{r-1} x_{s+r} = \partial^E x_{s+2} = \bar{\eta}_*^E x^E \qquad \text{and} \qquad f_{s+r-2*} \circ (\partial^E)^2 (x_{s+r}) = (\partial^F)^2 y_{s+r}.$$

Hence $\eta_*^E x_{s+r} \in \{d_r^E x^E\}$. Moreover, if $f_* \circ \lambda_* : E_2^{s+r-2,t+r-2}(X) \to F_2^{s+r-2,t+r-2}(Y)$ is epimorphic then $f_{s+r-1*} \circ \partial^E(x_{s+r}) = \partial^F y_{s+r}$, and so $f_* \circ \lambda_*(\eta_*^E x_{s+r}) = \eta_*^F \circ f_{s+r*}(x_{s+r}) = \eta_*^F y_{s+r} = y^F \in F_2^{s+r,t+r-1}(Y)$. □

Corollary 3.4. *For a map* $f : X \to Y$ *and two integers* t, s, *we assume the following:*

i) *There exists an integer* $r_0(s,t) \leq \infty$ *such that* $\bar{Z}E_2^{s+r,t+r-1}(X) = 0$ *for* $r > r_0(s,t)$.

ii) $f_* \circ \lambda_* : \bar{Z}E_2^{s+r,t+r-1}(X) \to \bar{Z}F_2^{s+r,t+r-1}(Y)$ *is monomorphic for any* $2 \leq r \leq r_0(s,t)$.

iii) $f_* \circ \lambda_* : E_2^{s+r,t+r}(X) \to F_2^{s+r,t+r}(Y)$ *is epimorphic for* $0 \leq r \leq r_0(s,t) - 2$.

Then the induced homomorphism $f_* \circ \lambda_* : ZE_2^{s,t}(X) \to ZF_2^{s,t}(Y)$ *is epimorphic.*

Proof. Take any element $y^F \in ZF_2^{s,t}(Y)$. By the assumption iii) for $r = 0$, we have $x^E \in E_2^{s,t}(X)$ with $f_* \circ \lambda_*(x^E) = y^F$. We notice that $\{d_r^F y^F\} \neq \emptyset$ for any $r \geq 2$ by Corollary 2.6 iii). Then Corollary 3.3 implies $\{d_r^E x^E\} \neq \emptyset$ for $r = r_0(s,t) + 1$. Now if $\{d_r^E x^E\} \neq \emptyset$ for $r > r_0(s,t)$ then $\{d_r^E x^E\} = \{0\}$ by the assumption i), and so $\{d_{r+1}^E x^E\} \neq \emptyset$ by Corollary 2.6 iii). By induction, we see that $\{d_r^E x^E\} \ni 0$ for any $r \geq 2$. Hence $x^E \in ZE_2^{s,t}(X)$ by Corollary 2.6 iii). We complete the proof of this corollary. □

Now we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Take any element $x \in \pi_t(X)$ with $f_*(x) = 0 \in \pi_t(Y)$. By the assumption ii) for $s = 0$, $\eta_*^E(x) = 0 \in E_*(X)$, and so we have $x_1 \in \pi_{t+1}(X_1^E)$ with $\partial^E x_1 = x$. By Lemma 3.2, there exists an element $x_{s_0(t)+1} \in \pi_{t+s_0(t)+1}(X_{s_0(t)+1}^E)$ such that $(\partial^E)^{s_0(t)+1}(x_{s_0(t)+1}) = \partial^E x_1 = x$. Hence $x = 0$ by Proposition 2.4 iv). □

Proof of Theorem 1.2. Take any element $y \neq 0 \in \pi_t(Y)$. By the assumption, we have an element $y_s \in \pi_{t+s}(Y_s^F)$ such that $(\partial^F)^s y_s = y$. We assume that there is no element $y' \in \pi_{t+s'}(Y_{s'}^F)$ such that $(\partial^F)^{s'} y' = y$ if $s' > s$. Then $s \leq s_1(t)$ by the assumption i) and Proposition 2.4 iv). By the assumption ii), we have an element $x_s \in \pi_{t+s}(X_s^E)$ with

$$f_* \circ \lambda_*(\eta_*^E(x_s)) = \eta_*^F(y_s) \in F_2^{s,t+s}(Y),$$

and so there exists an element $y_{s+1} \in \pi_{s+t+1}(Y_{s+1}^F)$ with $(\partial^F)^2 y_{s+1} = \partial^F\{y_s - f_{s*}(x_s)\}$ by Lemma 2.3 iii). Inductively we can take elements $y_{s'} \in \pi_{t+s'}(Y_{s'}^F)$ for $s \leq s' \leq s_1(t)+1$ and $x_{s'} \in \pi_{t+s'}(X_{s'}^F)$ for $s \leq s' \leq s_1(t)$ such that $(\partial^F)^2 y_{s'} = \partial^F\{y_{s'-1} - f_{s'-1*}(x_{s'-1})\}$. By the assumption i) and Proposition 2.4 iv), we see $(\partial^F)^{s_1(t)+1}(y_{s_1(t)+1}) = 0$. Now

$$\begin{aligned} f_* \sum_{s'=s}^{s_1(t)} (\partial^F)^{s'}(x_{s'}) &= \sum_{s'=s}^{s_1(t)} \{(\partial^F)^{s'} y_{s'} - (\partial^F)^{s'+1} y_{s'+1}\} \\ &= (\partial^F)^s y_s - (\partial^F)^{s_1(t)+1} y_{s_1(t)+1} = y. \end{aligned}$$

Hence f_* is epimorphic. □

Finally, we prove Theorems 1.3 and 1.4.

The ring map $\lambda : E \to F$ induces a map $L_\lambda : L_E X \to L_F L_E X = L_F X$ and maps between the E- and F-Adams spectral sequences

$$\begin{array}{ccc} \lambda_* : \{E_r^{s,t}(X)\} & \longrightarrow & \{F_r^{s,t}(X)\} \\ \cong \downarrow & & \downarrow \cong \\ L_{\lambda *} \circ \lambda_* : \{E_r^{s,t}(L_E X)\} & \longrightarrow & \{F_r^{s,t}(L_F X)\} \end{array}$$

with

$$E_2^{s,t}(X) = E_2^{s,t}(L_E X) \qquad \text{and} \qquad F_2^{s,t}(X) = F_2^{s,t}(L_F X).$$

We notice that

$$\begin{array}{ccccc} E_2^{s,t}(X) & \supset & ZE_2^{s,t}(X) & \supset & \bar{Z}E_2^{s,t}(X) \\ \cong \downarrow & & \cong \downarrow & & \downarrow \cap \\ E_2^{s,t}(L_E X) & \supset & ZE_2^{s,t}(L_E X) & \supset & \bar{Z}E_2^{s,t}(L_E X). \end{array} \tag{3.3}$$

If $\{E_r^{s,t}(X)\}$ converges to $\pi_*(L_E X)$, then

$$ZE_2^{s,t}(X) = ZE_2^{s,t}(L_E X) = \bar{Z}E_2^{s,t}(L_E X) \supset \bar{Z}E_2^{s,t}(X). \tag{3.4}$$

The same results hold for $F_2^{s,t}(X)$ and $F_2^{s,t}(L_F X)$.

Proof of Theorem 1.3. By Theorem 2.7 for $a = s_0(t)$, $b = t$, the assumptions ii–iii) of Theorem 1.3 imply that $\lambda_* : E_2^{s,u+s}(X) \to F_2^{s,u+s}(X)$ is monomorphic for $s \le s_0(t)$, $u = t$ and epimorphic for $s < s_0(t)$, $u = t+1$. Hence $L_{\lambda *} \circ \lambda_* : \bar{Z}E_2^{s,t+s}(L_E X) \to \bar{Z}F_2^{s,t+s}(L_F X)$ is monomorphic for $0 \le s \le s_0(t)$ and $L_{\lambda *} \circ \lambda_* : E_2^{s,t+s+1}(L_E X) \to F_2^{s,t+s+1}(L_F X)$ is epimorphic for $0 \le s \le s_0(t) - 2$ by (3.3). Now Theorem 1.1 implies this theorem. □

Proof of Theorem 1.4. By Theorem 2.7 for $a = s_1(t) + 1$, $b = t - 1$, the assumptions ii–iii) of Theorem 1.4 imply that $\lambda_* : E_2^{s',u+s'}(X) \to F_2^{s',u+s'}(X)$ is monomorphic for $s' \le s_1(t) + 1$, $u = t - 1$ and epimorphic for $s' < s_1(t) + 1$, $u = t$. We fix an integer $s \le s_1(t)$. Then $\lambda_* : \bar{Z}E_2^{s+r,t+s+r-1}(X) \to \bar{Z}F_2^{s+r,t+s+r-1}(X)$ is monomorphic for $0 \le r \le s_1(t) - s + 1$ and $L_{\lambda *} \circ \lambda_* : E_2^{s+r,t+s+r}(X) \to F_2^{s+r,t+s+r}(X)$ is epimorphic for $0 \le r \le s_1(t) - s$.

By the assumption i) of Theorem 1.4, $\bar{Z}E_2^{s',t+s'-1}(X) = 0$ for $s' > s_1(t) + 1$, and so $\bar{Z}E_2^{s+r,t+s+r-1}(X) = 0$ for $r > s_1(t) - s + 1$. Taking $r_0(s, t+s) = s_1(t) - s + 1$ in Corollary 3.4, we see that $\lambda_* : ZE_2^{s,t+s}(X) \to ZF_2^{s,t+s}(X)$ is epimorphic for $s \le s_1(t)$, and so is $L_{\lambda *} \circ \lambda_* : \bar{Z}E_2^{s,t+s}(L_E X) \to \bar{Z}F_2^{s,t+s}(L_F X)$ by (3.4). Now Theorem 1.2 implies this theorem. □

Proof of Corollary 1.5. We have split exact sequences

$$0 \to E_*(S^0 \wedge \bar{E}^n \wedge X) \to E_*(E \wedge \bar{E}^n \wedge X) \to E_*(\bar{E} \wedge \bar{E}^n \wedge X) \to 0.$$

By induction on n, $E_*(\bar{E}^n \wedge X)$ is flat over E_*. Now, $\mathrm{Ext}_{F_*F}^{s,*}(F_*, F_*(E \wedge \bar{E}^n \wedge X))$ is a cohomology group of a cochain complex

$$\{F_*F \otimes_{F_*} \cdots \otimes_{F_*} F_*F \otimes_{F_*} F_*(E \wedge \bar{E}^n \wedge X)\}$$
$$= \{F_*F \otimes_{F_*} \cdots \otimes_{F_*} F_*F \otimes_{F_*} F_*(E) \otimes_{E_*} E_*(\bar{E}^n \wedge X)\}.$$

Since F_*F is flat,

$$F_2^{s,*}(E \wedge \bar{E}^n \wedge X) = \mathrm{Ext}_{F_*F}^{s,*}(F_*, F_*(E \wedge \bar{E}^n \wedge X))$$
$$= \mathrm{Ext}_{F_*F}^{s,*}(F_*, F_*(E)) \otimes_{E_*} E_*(\bar{E}^n \wedge X)$$

by Remark 2.1 (see [4, (3.8.7–9)]). Hence $F_2^{s,*}(E \wedge \bar{E}^n \wedge X) = 0$ for $0 < s < s_0$ and

$$\phi^F : \pi_*(E \wedge \bar{E}^n \wedge X) \to \mathrm{Ext}_{F_*F}^{0,*}(F_*, F_*(E \wedge \bar{E}^n \wedge X))$$

is isomorphic. Now Theorems 1.3 and 1.4 imply this corollary. □

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