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Differentiability properties of some nonlinear operators associated to the conformal welding of Jordan curves in Schauder spaces

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Abstract. As it is well-known, to a given plane simple closed curve ζ with nonvanishing tangent vector, one can associate a conformal welding homeomorphism $\mathbf{w}[\zeta]$ of the unit circle to itself, obtained by composing the restriction to the unit circle of a suitably normalized Riemann map of the domain exterior to ζ with the inverse of the restriction to the unit circle of a suitably normalized Riemann map of the domain interior to ζ. Now we think the functions ζ and $\mathbf{w}[\zeta]$ as points in a Schauder function space on the unit circle, and we show that the correspondence $\mathbf{w}$ which takes ζ to $\mathbf{w}[\zeta]$ is real differentiable for suitable exponents of the Schauder spaces involved. Then we show that $\mathbf{w}$ has a right inverse which is the restriction of a holomorphic nonlinear operator.

1. Introduction

As it is well-known, given an element ζ of the set $\mathscr{A}_{\partial\mathbf{D}}$ of the complex-valued differentiable injective functions, with nonvanishing first derivative, defined on the boundary $\partial\mathbf{D}$ of the open unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$, the function ζ parametrizes a Jordan curve. To each $\zeta \in \mathscr{A}_{\partial\mathbf{D}}$, one can associate a pair (G, F) of Riemann maps, with G a suitably normalized holomorphic homeomorphism of the exterior $\mathbf{C}\backslash \mathrm{cl}\,\mathbf{D}$ of $\mathbf{D}$ onto the exterior $E[\zeta]$ of ζ, and with F a suitably normalized holomorphic homeomorphism of $\mathbf{D}$ onto the interior $I[\zeta]$ of ζ. It is also well-known that G and F can be extended with continuity to boundary homeomorphisms. Thus one can consider the so-called conformal welding homeomorphism $F^{(-1)} \circ G_{|\partial\mathbf{D}}$ of $\partial\mathbf{D}$, which we denote by $\mathbf{w}[\zeta]$. Now let $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$ be the Schauder space of m-times continuously differentiable complex-valued functions on $\partial\mathbf{D}$, whose m-th order derivative is α-Hölder continuous, with $\alpha \in]0,1[$, $m \geq 1$. It is well-known that if $\zeta \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, then $\mathbf{w}[\zeta] \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. In this paper we first prove some differentiability theorems for the nonlinear 'conformal welding operator' $\mathbf{w}[\cdot]$. We note that such theorems can be shown to be optimal in the frame of Schauder spaces (cf. [19, Thm. 2.14].) Moreover,

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we observe that by restricting $\mathbf{w}[\cdot]$ to the set of ζ's which are boundary values of Riemann maps defined on $\mathbf{C}\backslash\mathbf{D}$, the operator $\mathbf{w}[\cdot]$ becomes real analytic.

Next we turn to the problem of constructing a right inverse of $\mathbf{w}[\cdot]$. The problem of constructing a suitably normalized pair of functions (G, F) as above, such that $F^{(-1)} \circ G_{|\partial\mathbf{D}} = \phi$ by a given regular orientation preserving homeomorphism ϕ of $\partial\mathbf{D}$ to itself, a so-called 'shift' of $\partial\mathbf{D}$, is known as the conformal sewing problem and is a particular type of boundary value problem with shift for sectionally holomorphic functions. By exploiting a classical method (cf. *e.g.*, Lu [22]), one can show that to each shift $\phi \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, one can associate a unique suitably normalized pair of functions (G, F) as above. Then the nonlinear operator $\mathbf{s}$, which takes ϕ to $\mathbf{s}[\phi] \equiv G_{|\partial\mathbf{D}}$ is a right inverse of $\mathbf{w}$, and will be called the 'conformal sewing operator'. Next we prove that $\mathbf{s}[\phi] \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$ if the shift $\phi \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. Then we analyze the differentiability properties of $\mathbf{s}$. Since the domain of $\mathbf{s}$, namely the set of positively oriented $\phi \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ such that $\phi(\partial\mathbf{D}) = \partial\mathbf{D}$ is not open in the Banach space $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$, we construct an extension of $\mathbf{s}$ to the open set of orientation preserving elements of $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, and we show that such extension is complex-analytic. In other words, we show that the boundary values of the Riemann map G of the domain exterior to the curve $\mathbf{s}[\phi]$, depend complex-analytically on ϕ. Then we consider the Riemann map F which is related to G by the equality $F_{|\partial\mathbf{D}} = G \circ \phi^{(-1)} = \mathbf{s}[\phi] \circ \phi^{(-1)}$. We deduce the differentiability properties of the dependence of the boundary values of F upon ϕ by 'ad hoc' variants of the differentiability results on the inversion and on the composition operator of [15]. We note that the differentiability results for the dependence of F on ϕ can be shown to be sharp by means of inverse theorems. In particular, one can show that F does not depend complex analytically on ϕ (cf. [19, Thm. 2.17].)

The theory of boundary value problems with shift for sectionally holomorphic functions, also called Haseman problems, is well-known and started with Haseman [9]. Kveselava [13] developped an existence and uniqueness theory in case ϕ is of class $C_*^{1,\alpha}$. Later, other Haseman type problems have been studied, also for more general shifts (cf. Litvinchuk [21], Monakhov [23, pp. 357–367].) In the direction of the perturbation results however, the authors are only aware of the continuity result for the conformal welding operator of David [4], and of the continuity results for the conformal sewing operator of Monakhov [23, p. 363], and of Huber and Kühnau [11], in different function space settings. We mention also the work of Nag [24], who has considered a one-parameter family $\{\phi_t\}$ of shifts depending real analytically on a real parameter t, and who has provided an algorithm to compute the coefficients of the formal expansion of the corresponding families of curves $\mathbf{s}[\phi_t]$ and $\mathbf{s}[\phi_t] \circ \phi_t^{(-1)}$, under the assumption that such expansions converge.

We believe that our results could be employed in the perturbation analysis of other well-posed Haseman problems. Indeed, the operator which maps a shift to the corresponding solution of the Haseman problem can be expressed in terms of the conformal sewing operator and of operators of known regularity (cf. *e.g.*, Gakhov [6, p. 129, § 14].)

This paper is organized as follows. Section 2 is a section of preliminaries and notation. Section 3 concerns the definition of the conformal welding map and contains differentiability theorems for the conformal welding operator. Section 4 is devoted to the definition of the conformal sewing operator and of its extension. Section 5 contains a complex differentiability theorem for the conformal sewing operator.

2. Technical preliminaries and notation

Let $\mathscr{X}$, $\mathscr{Y}$ be normed spaces over the field $\mathbf{K}$, with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. We say that $\mathscr{X}$ is continuously imbedded in $\mathscr{Y}$ provided that $\mathscr{X} \subseteq \mathscr{Y}$ and that the inclusion map is continuous. We say that a map T of a subset of $\mathscr{X}$ to $\mathscr{Y}$ is compact, provided that it maps bounded sets to sets with compact closure. For standard definitions of Calculus in normed spaces, we refer to Prodi and Ambrosetti [28] or to Berger [2]. Unless otherwise specified, we understand that a finite product of normed spaces is endowed with the supremum of the norms of the components. Let $\mathbf{N}$ be the set of nonnegative integers including zero. Throughout the paper, n denotes an element of $\mathbf{N}\backslash\{0\}$. A complex normed space can be viewed naturally as a real normed space. Accordingly, we will say that a certain map between complex normed spaces is real linear, real differentiable, or real analytic, to indicate that such map is linear, differentiable or analytic between the corresponding underlying real spaces, respectively. To emphasize that we are retaining the complex structure, we will say that the map is complex linear, complex differentiable, or complex analytic, respectively. The inverse function of a function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex valued function g, which is denoted g^{-1}. For all subsets B of $\mathbf{R}^n$, the closure of B is denoted $\mathrm{cl}\, B$. We now define the Schauder spaces on the closure of an open subset of $\mathbf{R}^n$. Let Ω be an open subset of $\mathbf{R}^n$, $m \in \mathbf{N}$. We denote by $C^m(\Omega, \mathbf{C})$ the space of m-times continuously real-differentiable complex-valued functions on Ω, and by $C^m(\mathrm{cl}\,\Omega, \mathbf{C})$ the subspace of those functions of $C^m(\Omega, \mathbf{C})$ such that for all $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbf{N}^n$, with $|\eta| \equiv \eta_1 + \cdots + \eta_n \leq m$, the function $D^\eta f \equiv \frac{\partial^{|\eta|} f}{\partial^{\eta_1} x_1 \ldots \partial^{\eta_n} x_n}$ can be extended with continuity to $\mathrm{cl}\,\Omega$. If Ω is bounded, then $C^m(\mathrm{cl}\,\Omega, \mathbf{C})$ endowed with the norm defined by $\|f\|_{C^m(\mathrm{cl}\,\Omega, \mathbf{C})} \equiv \sum_{|\eta| \leq m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. If Ω is bounded and if $\alpha \in]0, 1]$, we denote by $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$ the subspace of $C^m(\mathrm{cl}\,\Omega, \mathbf{C})$ of those functions which have α-Hölder continuous derivatives of order m. If $f \in C^{0,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$, then we set $|f : \Omega|_\alpha \equiv \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in \mathrm{cl}\,\Omega, x \neq y\right\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$ is endowed with its usual norm $\|f\|_{C^{m,\alpha}(\mathrm{cl}\,\Omega,\mathbf{C})} \equiv \sum_{|\eta|\leq m} \sup_{\mathrm{cl}\,\Omega}|D^\eta f| + \sum_{|\eta|=m} |D^\eta f : \Omega|_\alpha$, and it is well-known to be a Banach space. If $B \subseteq \mathbf{C}$, then $C^{m,\alpha}(\mathrm{cl}\,\Omega, B)$ denotes the set $\{f \in C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C}) : f(\mathrm{cl}\,\Omega) \subseteq B\}$. By $H(\Omega)$ we understand the space of holomorphic functions of Ω to $\mathbf{C}$. Finally, the space $C^{m,\alpha,0}(\mathrm{cl}\,\Omega, \mathbf{C})$ is defined as the closure of $C^\infty(\mathrm{cl}\,\Omega, \mathbf{C})$ in $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$. Then we have the following.

Lemma 2.1. *Let* $m \in \mathbf{N}$, $\alpha \in]0,1]$. *Let* Ω *be a bounded open connected subset of* $\mathbf{R}^n$ *of class* C^{m+1}. *Then* $C^{m,\alpha,0}(\mathrm{cl}\,\Omega, \mathbf{C})$ *coincides with the closure in* $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$ *of the set of restrictions to* $\mathrm{cl}\,\Omega$ *of the polynomials with complex coefficients in n real variables. Moreover,* $C^{m,\alpha,0}(\mathrm{cl}\,\Omega, \mathbf{C})$ *contains* $C^{m+1}(\mathrm{cl}\,\Omega, \mathbf{C})$ *and* $C^{m,\beta}(\mathrm{cl}\,\Omega, \mathbf{C})$, *for all* $\beta \in]\alpha, 1]$.

Proof. Since Ω is of class C^{m+1}, then all functions of $C^{m+1}(\mathrm{cl}\,\Omega, \mathbf{C})$ are restrictions of some element of $C^{m+1}(\mathbf{R}^n, \mathbf{C})$ (cf. *e.g.*, Troianiello [30, p. 13].) Then by Weierstrass Theorem (cf. *e.g.*, Rohlin and Fuchs [29, p. 185]), all elements of $C^{m+1}(\mathbf{R}^n, \mathbf{C})$ can be approximated in the $C^{m+1}(\mathrm{cl}\,\Omega, \mathbf{C})$-norm by polynomials. Since $\mathrm{cl}\,\Omega$ is of class C^{m+1}, then $C^{m+1}(\mathrm{cl}\,\Omega, \mathbf{C})$ is continuously imbedded in $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$ (cf. *e.g.*, [15, p. 460].) Then the first part of the statement and the inclusion $C^{m+1}(\mathrm{cl}\,\Omega, \mathbf{C}) \subseteq C^{m,\alpha,0}(\mathrm{cl}\,\Omega, \mathbf{C})$ follow. Now let $f \in C^{m,\beta}(\mathrm{cl}\,\Omega, \mathbf{C})$. Since Ω is of class C^{m+1}, then f admits an extension of class $C^{m,\beta}$ and with compact support in a ball containing $\mathrm{cl}\,\Omega$ (cf. *e.g.*, Troianiello [30, Thm. 1.3, p. 13].) By taking the convolution with a family of mollifiers, such extension can be approximated by a sequence of C^∞ functions bounded in $C^{m,\beta}(\mathrm{cl}\,\Omega, \mathbf{C})$ and convergent in $C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{C})$ (cf. *e.g.*, Troianiello [30, pp. 20, 21].) Then $f \in C^{m,\alpha,0}(\mathrm{cl}\,\Omega, \mathbf{C})$. □

We now define the Schauder spaces on plane Jordan curves, which are particular compact subsets of $\mathbf{C}$ with no isolated points. With somewhat more generality, we define the Schauder spaces on a general compact subset K of $\mathbf{C}$ with no isolated points. We say that a function f of K to $\mathbf{C}$ is complex differentiable at $z_0 \in \mathbf{C}$ if $\lim_{K \ni z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists finite. We denote such limit by $f'(z_0)$. As usual the higher order derivatives, if they exist, are defined inductively. Let $m \in \mathbf{N}$. We denote by $C_*^m(K, \mathbf{C})$ the complex normed space of the m-times continuously complex differentiable functions f of K to $\mathbf{C}$ endowed with the norm $\|f\|_{C_*^m(K,\mathbf{C})} = \sum_{j=0}^m \sup_K|f^{(j)}|$. If $\alpha \in]0,1]$, we denote by $C_*^{m,\alpha}(K, \mathbf{C})$ the subspace of $C_*^m(K, \mathbf{C})$ of those functions having α-Hölder continuous m-th order derivative in K. If $f \in C_*^{0,\alpha}(K, \mathbf{C})$, then we set $|f : K|_\alpha \equiv \sup\left\{\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha} : z_1, z_2 \in K, z_1 \neq z_2\right\}$. We endow $C_*^{m,\alpha}(K, \mathbf{C})$ with the norm $\|f\|_{C_*^{m,\alpha}(K,\mathbf{C})} \equiv \|f\|_{C_*^m(K,\mathbf{C})} + |f^{(m)} : K|_\alpha$. If $B \subseteq \mathbf{C}$, we set $C_*^{m,\alpha}(K, B) \equiv$

$\{f \in C_*^{m,\alpha}(K, \mathbf{C}) : f(K) \subseteq B\}$. We denote by $C_*^{m,\alpha,0}(K, \mathbf{C})$ the closure of $C_*^\infty(K, \mathbf{C})$ in $C_*^{m,\alpha}(K, \mathbf{C})$. Then the following variant of [14, Cor. 4.24, Prop. 4.29] holds (cf. [18, Lem. 2.5].)

Lemma 2.2. *The following statements hold.*

(i) *Let* $\phi \in C_*^1(\partial\mathbf{D}, \mathbf{C})$. *Then* $l_{\partial\mathbf{D}}[\phi] \equiv \inf\left\{\frac{|\phi(x)-\phi(y)|}{|x-y|} : x, y \in \partial\mathbf{D}, x \neq y\right\} > 0$ *if and only if* ϕ *is injective and* $\phi'(\xi) \neq 0$ *for all* ξ *in* $\partial\mathbf{D}$.

(ii) *The function of* $C_*^1(\partial\mathbf{D}, \mathbf{C})$ *to* $\mathbf{R}$ *which maps* ϕ *to* $l_{\partial\mathbf{D}}[\phi]$ *is continuous, and in particular, the set* $\mathscr{A}_{\partial\mathbf{D}} \equiv \{\phi \in C_*^1(\partial\mathbf{D}, \mathbf{C}) : l_{\partial\mathbf{D}}[\phi] > 0\}$ *is open in* $C_*^1(\partial\mathbf{D}, \mathbf{C})$.

(iii) $\min_{x\in\partial\mathbf{D}} |\phi'(x)| \geq l_{\partial\mathbf{D}}[\phi]$, *for all* $\phi \in C_*^1(\partial\mathbf{D}, \mathbf{C})$.

We are now ready to state the following, which collects a few facts which we need on the spaces $C_*^{m,\alpha}(K, \mathbf{C})$. For a proof and for appropriate references, we refer to [18, Lems. 2.7, 2.8].

Lemma 2.3. *Let* $m \in \mathbf{N}$, $\alpha, \beta \in]0, 1]$, $\phi \in \mathscr{A}_{\partial\mathbf{D}}$, $L = \phi(\partial\mathbf{D})$. *Then the following statements hold.*

(i) $C_*^{m+1}(L, \mathbf{C})$ *is continuously imbedded in* $C_*^{m,\alpha}(L, \mathbf{C})$. *If* $\alpha < \beta$, *then* $C_*^{m,\beta}(L, \mathbf{C})$ *is compactly imbedded in* $C_*^{m,\alpha}(L, \mathbf{C})$.

(ii) *The pointwise product is continuous in the Banach space* $C_*^{m,\alpha}(L, \mathbf{C})$.

(iii) *The reciprocal map in* $C_*^{m,\alpha}(L, \mathbf{C})$, *which maps a nonvanishing function* f *to its reciprocal, is complex analytic from* $C_*^{m,\alpha}(L, \mathbf{C}\setminus\{0\})$ *to itself.*

(iv) *Let* $\phi_1 \in \mathscr{A}_{\partial\mathbf{D}}$, $L_1 = \phi_1(\partial\mathbf{D})$. *If* $f \in C_*^{m,\alpha}(L_1, \mathbf{C})$ *and if* $g \in C_*^{m,\beta}(L, L_1)$, *then* $f \circ g \in C_*^{m,\gamma_m(\alpha,\beta)}(L, \mathbf{C})$ *with* $\gamma_0(\alpha, \beta) = \alpha\beta$ *and* $\gamma_m(\alpha, \beta) = \min\{\alpha, \beta\}$ *if* $m > 0$.

(v) *Let* $m \geq 1$. *If* $g \in C_*^{m,\alpha}(L, \mathbf{C})$ *is injective and satisfies condition* $g'(\xi) \neq 0$, *for all* $\xi \in L$, *then* $g^{(-1)} \in C_*^{m,\alpha}(g(L), L)$.

(vi) *If* $I[\phi]$ *and* $E[\phi]$ *denote the bounded and the unbounded open connected component of* $\mathbf{C}\setminus\phi(\partial\mathbf{D})$, *respectively, then* $\partial I[\phi] = \partial E[\phi] = \phi(\partial\mathbf{D})$.

(vii) *If* $f \in \mathscr{A}_{\partial\mathbf{D}}$, *and if* $f(\partial\mathbf{D}) \subseteq \partial\mathbf{D}$, *then* $f(\partial\mathbf{D}) = \partial\mathbf{D}$ *and* f *is a homeomorphism of* $\partial\mathbf{D}$ *to itself.*

We now introduce two differentiability theorems, for the composition and for the inversion operator. To do so, we need the following, which we use to study the regularity of the operator $\mathbf{w}[\cdot]$, and the regularity of the dependence of F on the shift ϕ.

Lemma 2.4. *Let* $m \in \mathbf{N}$, $\alpha \in]0, 1]$, $R \in]1, +\infty[$. *Let* $R\mathbf{D} \equiv \{x \in \mathbf{R}^2 : |x| < R\}$. *Then there exists a linear and continuous extension operator* $\mathbf{E}$ *of* $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$ *to* $C^{m,\alpha}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ *such that the following statements hold.*

(i) $(\mathbf{E}[f])_{|\partial\mathbf{D}} = f$, *for all* $f \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$, *and* $\mathbf{E}[f] \in C^{m,\alpha,0}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ *for all* $f \in C_*^{m,\alpha,0}(\partial\mathbf{D}, \mathbf{C})$.

(ii) *Let* $1 \leq j \leq m$. *For all* $f \in C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$, *the real differential of order* j *of the function* $\mathbf{E}[f]$ *at* $\tau \in \partial \mathbf{D}$ *satisfies the following equation*

$$d^j \mathbf{E}[f](\tau)(\sigma_1, \dots, \sigma_j) = f^{(j)}(\tau)\sigma_1 \dots \sigma_j, \tag{2.5}$$

for all $(\sigma_1, \dots, \sigma_j) \in \mathbf{C}^j$. *In particular,* $d^j \mathbf{E}[f](\tau)$ *is also a complex* j*-multilinear operator, whenever* $\tau \in \partial \mathbf{D}$.

Proof. To prove statement (i), we first show that there exists a linear and continuous operator Z of $\prod_{l=0}^m C_*^{m-l,\alpha}(\partial \mathbf{D}, \mathbf{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$ such that $\left(Z[\mathbf{f}]_{|\partial \mathbf{D}}, \dots, \frac{\partial^m Z[\mathbf{f}]}{\partial \nu^m}_{|\partial \mathbf{D}}\right) = \mathbf{f}$ for all $\mathbf{f} \in \prod_{l=0}^m C_*^{m-l,\alpha}(\partial \mathbf{D}, \mathbf{C})$, where ν is the outer unit normal to $\partial \mathbf{D}$. If $r, s \in \{0, \dots, m\}$, then we set $\delta_{rs} = 1$ if $r = s$, $\delta_{rs} = 0$ if $r \neq s$. As a first step we fix an arbitrary $l \in \{0, \dots, m\}$, and we show the existence of a linear and continuous operator Z_l of $C_*^{m-l,\alpha}(\partial \mathbf{D}, \mathbf{C})$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$ such that $\frac{\partial^j Z_l[h]}{\partial \nu^j}_{|\partial \mathbf{D}} = \delta_{jl} l! h$ for $0 \leq j \leq l$, and for all $h \in C_*^{m-l,\alpha}(\partial \mathbf{D}, \mathbf{C})$. By a standard argument based on the partition of unity and on the use of local charts for $\partial \mathbf{D}$, the existence of Z_l follows from that of a linear and continuous operator $\tilde{Z}_l$ of $C^{m-l,\alpha}([-1,1], \mathbf{C})$ to $C^{m,\alpha}(\mathrm{cl}(]-1, 1[\times]-1,0[), \mathbf{C})$ such that $\frac{\partial^j \tilde{Z}_l[g]}{\partial x_2^j}_{|x_2=0} = \delta_{jl} l! g$ for $0 \leq j \leq l$. Let K be a linear and continuous operator of $C^{m-l,\alpha}([-1,1], \mathbf{C})$ to $C^{m-l,\alpha}([-2,2], \mathbf{C})$ with $K[g] = g$ on $[-1,1]$ and $\mathrm{supp}\, K[g] \subseteq]-2,2[$, for all $g \in C^{m-l,\alpha}([-1,1], \mathbf{C})$. Furthermore, one can choose K so that K maps $C^{m-l+1}([-1,1], \mathbf{C})$ to $C^{m-l+1}([-2,2], \mathbf{C})$ (cf. *e.g.*, the construction of Troianiello [30, Thm. 1.3, p. 13] with $k = m - l + 1$.) To construct $\tilde{Z}_l$, we take $l+1$ distinct real numbers $\alpha_0, \dots, \alpha_l$, and we determine $\beta_0, \dots, \beta_l$ by solving the (Vandermonde) system $\sum_{s=0}^l \alpha_s^j \beta_s = \delta_{jl} l!$, $j = 0, \dots, l$, and we set $\tilde{Z}_l[g](x_1, x_2) \equiv \sum_{s=0}^l \beta_s G_l[g](x_1 + \alpha_s x_2)$, where $G_l[g]$ is the m times differentiable function of $\mathbf{R}$ to $\mathbf{C}$ determined by conditions $\frac{d^l}{dt^l} G_l[g] = K[g]$, $\frac{d^j}{dt^j}_{|t=0} G_l[g] = 0$ for $0 \leq j < l$. Then one can define Z by exploiting the operators Z_l and formula (5.8) of Nečas [25, p. 93]. It is also clear that Z maps $\prod_{l=0}^m C_*^{m+1-l}(\partial \mathbf{D}, \mathbf{C})$ to $C^{m+1}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$. Since $\mathrm{cl}\,\mathbf{D}$ is of class C^∞, it is also known that there exists a linear and continuous extension operator $\mathbf{E}_R$ of $C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$ to $C^{m,\alpha}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ such that $\mathbf{E}_R[v]_{|\mathrm{cl}\,\mathbf{D}} = v$, for all $v \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$. Furthermore, one can choose $\mathbf{E}_R$ so that $\mathbf{E}_R$ maps $C^{m+1}(\mathrm{cl}\,\mathbf{D}, \mathbf{C})$ to $C^{m+1}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ (cf. *e.g.*, the construction of Troianiello [30, Thm. 1.3, p. 13] with $k = m + 1$.) Then we set $\mathbf{E}[f] \equiv \mathbf{E}_R \circ Z[f(\tau), \tau f'(\tau), \dots, \tau^m f^{(m)}(\tau)]$. If $f \in C_*^\infty(\partial \mathbf{D}, \mathbf{C})$, then $\mathbf{E}[f] \in C^{m+1}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ and thus $\mathbf{E}[f] \in C^{m,\alpha,0}(\mathrm{cl}(R\mathbf{D}), \mathbf{C})$ by Lemma 2.1. We now prove (ii). By construction, the function $\mathbf{E}[f]$ is m-times real differentiable at each point $\tau \in \partial \mathbf{D}$, and the real differential $d^j \mathbf{E}[f](\tau)$ is a real j-multilinear operator of $\mathbf{R}^{2j}$ to $\mathbf{R}^2$. Also, the right hand side of equation (2.5) delivers a complex j-multilinear operator, which we denote by $M_{f^{(j)}(\tau)}$ of $\mathbf{C}^j$ to $\mathbf{C}$, and thus a real j-multilinear operator of $\mathbf{R}^{2j}$ to $\mathbf{R}^2$. In order to prove equality (2.5), it suffices to show that $d^j\mathbf{E}[f](\tau) = M_{f^{(j)}(\tau)}$ on the j-tuples of elements of a real basis of $\mathbf{R}^2$. To shorten our notation, we write $v^{[l]}$ instead of $\overbrace{v,\dots,v}^{l \text{ terms}}$ in the argument of a multilinear operator. Once $\tau \equiv (\tau_1, \tau_2) \in \partial\mathbf{D}$ is fixed, we choose $\{(\tau_1, \tau_2), (-\tau_2, \tau_1)\}$ as a real basis of $\mathbf{R}^2$. Note that τ equals the exterior unit normal to $\partial\mathbf{D}$ at τ, and that $i\tau = (-\tau_2, \tau_1)$ lies in the tangent space to $\partial\mathbf{D}$ at τ. Since $d^j\mathbf{E}[f](\tau)$ and $M_{f^{(j)}(\tau)}$ are multilinear and symmetric operators, it suffices to check that for $0 \le l \le j$, we have

$$d^j\mathbf{E}[f](\tau)((-\tau_2, \tau_1)^{[l]}, (\tau_1, \tau_2)^{[j-l]}) = f^{(j)}(\tau)(i\tau)^l \tau^{j-l}. \tag{2.6}$$

We now prove (2.6) by induction on $j \in \{1, \dots, m\}$. In case $j = 1$, it suffices to prove the following two equalities

$$d\mathbf{E}[f](\tau)((\tau_1, \tau_2)) = f'(\tau)\tau, \qquad d\mathbf{E}[f](\tau)((-\tau_2, \tau_1)) = f'(\tau)i\tau. \tag{2.7}$$

The first equality of (2.7) follows by equality $\frac{\partial}{\partial v}\mathbf{E}[f](\tau) = f'(\tau)\tau$, which holds by construction of $\mathbf{E}$. We now turn to prove the second equality of (2.7). We know that $\mathbf{E}[f](\cos\theta, \sin\theta) = f(e^{i\theta})$, for all $\theta \in [0, 2\pi]$. Then by differentiating with respect to θ, we obtain $d\mathbf{E}[f](\cos\theta, \sin\theta)((-\sin\theta, \cos\theta)) = f'(e^{i\theta})ie^{i\theta}$, which implies the validity of the second equation of (2.7). If $m = 1$, the proof is complete, thus we can assume that $m > 1$. We assume that equality (2.6) holds for $j \in \{1, \dots, m-1\}$, and for all $0 \le l \le j$, and we prove (2.6) for $j+1$, and for all $0 \le l \le j+1$. If $l = 0$, then (2.6) follows by equality $\frac{\partial^{j+1}}{\partial v^{j+1}}\mathbf{E}[f](\tau) = \tau^{j+1} f^{(j+1)}(\tau)$, which holds by construction of $\mathbf{E}[f]$. Thus we can assume that $l \ge 1$. By inductive assumption, we have

$$d^j\mathbf{E}[f](\tau)((-\tau_2, \tau_1)^{[l-1]}, (\tau_1, \tau_2)^{[j-l+1]}) = f^{(j)}(\tau)(i\tau)^{l-1}\tau^{j-l+1}. \tag{2.8}$$

Now by setting $\tau \equiv (\tau_1, \tau_2) = (\cos\theta, \sin\theta)$ in (2.8), and by differentiating with respect to θ, we obtain

$$\begin{aligned}
&d^{j+1}\mathbf{E}[f](\tau)((-\tau_2, \tau_1)^{[l]}, (\tau_1, \tau_2)^{[j-l+1]}) \\
&\qquad + (l-1)d^j\mathbf{E}[f](\tau)((-\tau_1, -\tau_2), (-\tau_2, \tau_1)^{[l-2]}, (\tau_1, \tau_2)^{[j-l+1]}) \\
&\qquad + (j-l+1)d^j\mathbf{E}[f](\tau)((-\tau_2, \tau_1)^{[l-1]}, (-\tau_2, \tau_1), (\tau_1, \tau_2)^{[j-l]}) \\
&\quad = f^{(j+1)}(\tau)(i\tau)^l\tau^{j-l+1} + f^{(j)}(\tau)(l-1)(i\tau)^{l-2}(-\tau)\tau^{j-l+1} \\
&\qquad + f^{(j)}(\tau)(i\tau)^{l-1}(j-l+1)\tau^{j-l}i\tau.
\end{aligned}$$

By exploiting the symmetry, the real j-multilinearity of $d^j\mathbf{E}[f](\tau)$, and the inductive assumption, we obtain that (2.6) holds for $j+1$, and for all $0 \le l \le j+1$. □

We now have the following variant of [15, Thm. 4.19]. See also Henry [10, p. 96]. For references to previous contributions on this issue by various authors, we refer to [15].

Theorem 2.9. *Let $m, r \in \mathbf{N}$, $\alpha, \beta \in]0,1]$. Let $\gamma_m(\alpha,\beta)$ be defined as in Lemma* 2.3 (iv). *Let $R \in]1,+\infty[$. Let $\mathbf{E}$ be the extension operator of Lemma* 2.4. *The operator $\tilde{\mathbf{T}}$ from $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},R\mathbf{D})$ to $C_*^{m,\gamma_m(\alpha,\beta)}(\partial\mathbf{D},\mathbf{C})$ defined by setting $\tilde{\mathbf{T}}[f,g] \equiv (\mathbf{E}[f]) \circ g$, for all $(f,g) \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},R\mathbf{D})$ is of class C^r in the real sense. The restriction of $\tilde{\mathbf{T}}$ to $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},\partial\mathbf{D})$ coincides with the ordinary composition operator $\mathbf{T}$ defined by $\mathbf{T}[f,g] \equiv f \circ g$. The ordinary composition $\mathbf{T}$ maps $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta,0}(\partial\mathbf{D},\partial\mathbf{D})$ to $C_*^{m,\gamma_m(\alpha,\beta),0}(\partial\mathbf{D},\mathbf{C})$. If $r \geq 1$, $q \in \{1,\dots,r\}$, and if $(f_0,g_0) \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},\partial\mathbf{D})$, then the real differential of order q of $\tilde{\mathbf{T}}$ at (f_0,g_0) is delivered by the formula*

$$d^q\tilde{\mathbf{T}}[f_0,g_0]((v_{[1]},w_{[1]}),\dots,(v_{[q]},w_{[q]})) = \left(\sum_{j=1}^{q}(v_{[j]}^{(q-1)} \circ g_0)w_{[1]}\dots\widehat{w_{[j]}}\dots w_{[q]}\right) + (f_0^{(q)} \circ g_0)w_{[1]}\dots w_{[q]} \tag{2.10}$$

for all $((v_{[1]},w_{[1]}),\dots,(v_{[q]},w_{[q]})) \in (C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C}))^q$, where the '$\widehat{\ }$' symbol on a factor denotes that such factor should not appear in the product.

Proof. We first prove that $\tilde{\mathbf{T}}$ is of class C^r. It clearly suffices to show that given $(f^\#,g^\#) \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta}(\partial\mathbf{D},R\mathbf{D})$, the map $\tilde{\mathbf{T}}$ is of class C^r in an open neighborhood of $(f^\#,g^\#)$. Now we set $C_\varepsilon \equiv \{z \in \mathbf{C} : ||z|-1| < \varepsilon\}$ for all $\varepsilon > 0$. By uniform continuity of $\mathbf{E}[g^\#]$ on $\mathrm{cl}(R\mathbf{D})$ and by the inclusion $\mathbf{E}[g^\#](\partial\mathbf{D}) \subseteq R\mathbf{D}$, there exists $\varepsilon > 0$ such that $\mathbf{E}[g^\#](\mathrm{cl}\, C_\varepsilon) \subseteq R\mathbf{D}$. Clearly, $\mathscr{W}^\# \equiv \{g \in C_*^{m,\beta}(\partial\mathbf{D},R\mathbf{D}) : \mathbf{E}[g](\mathrm{cl}\, C_\varepsilon) \subseteq R\mathbf{D}\}$ is an open neighborhood of $g^\#$ in $C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$. By [16, Thm. 5.3] and Lemma 2.1, $\mathbf{T}$ is of class C^r from $C^{m+r,\alpha,0}(\mathrm{cl}(R\mathbf{D}),\mathbf{C}) \times C^{m,\beta}(\mathrm{cl}\, C_\varepsilon, R\mathbf{D})$ to $C^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\, C_\varepsilon,\mathbf{C})$. Furthermore, the restriction operator is easily seen to be linear and continuous from $C^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\, C_\varepsilon,\mathbf{C})$ to $C_*^{m,\gamma_m(\alpha,\beta)}(\partial\mathbf{D},\mathbf{C})$ (for example, by arguing as in [18, Lem. 2.8 (ii)].) Thus, $\tilde{\mathbf{T}}$ is of class C^r from $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times \mathscr{W}^\#$ to $C_*^{m,\gamma_m(\alpha,\beta)}(\partial\mathbf{D},\mathbf{C})$. Formula (2.10) follows by formula (2.5) and by the formula for the derivatives of $\mathbf{T}$ of [16, Rmk. 5.4]. By definition of the space $C^{m,\beta,0}(\mathrm{cl}\, C_\varepsilon,\mathbf{C})$ and by continuity of $\mathbf{T}$ from $C^{m,\alpha,0}(\mathrm{cl}(R\mathbf{D}),\mathbf{C}) \times C^{m,\beta}(\mathrm{cl}\, C_\varepsilon, R\mathbf{D})$ to $C^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\, C_\varepsilon,\mathbf{C})$, and by Lemma 2.4 (i), $\mathbf{T}$ maps $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \times C_*^{m,\beta,0}(\partial\mathbf{D},\partial\mathbf{D})$ to $C_*^{m,\gamma_m(\alpha,\beta),0}(\partial\mathbf{D},\mathbf{C})$. □

We now turn to the study of the inversion operator by showing the validity of the following variant of [15, Thm. 5.9].

Theorem 2.11. *Let* $m \in \mathbf{N} \setminus \{0\}$, $r \in \mathbf{N}$, $\alpha \in]0,1]$. *Let* $\mathbf{J}$ *be the operator of* $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$ *to* $C_*^{m,\alpha}(\partial\mathbf{D},\partial\mathbf{D})$ *defined by equality* $\mathbf{J}[f] \equiv f^{(-1)}$, *for all* $f \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$. *If* $r = 0$, *then* $\mathbf{J}$ *is continuous and the image of* $\mathbf{J}$ *is contained in* $C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$. *If* $r \geq 1$, *then for all* $f_0 \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$, *there exist an open neighborhood* $\mathscr{W}_{f_0}$ *of* f_0 *in* $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, *and an operator* $\tilde{\mathbf{J}}_{f_0}$ *of class* C^r *in the real sense from* $\mathscr{W}_{f_0}$ *to* $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, *such that*

$$\tilde{\mathbf{J}}_{f_0}[f] = \mathbf{J}[f], \qquad \forall f \in \mathscr{W}_{f_0} \cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}). \tag{2.12}$$

Furthermore, the following formula holds

$$d\tilde{\mathbf{J}}_{f_0}[f](w) = -(f' \circ f^{(-1)})^{-1} w \circ f^{(-1)}, \qquad \forall w \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}), \tag{2.13}$$

for all $f \in \mathscr{W}_{f_0} \cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$.

Proof. We first consider case $r = 0$. By Lemma 2.3 (v), (vii), the following inclusion holds $\mathbf{J}(C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}) \subseteq C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$. We now prove that $\mathbf{J}$ is continuous from $C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$ to $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$. To do so, we proceed by induction on m. Let $m = 1$, $\lim_{j\to\infty} f_j = f$ in $C_*^{1,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$. Since the sequence $\{f_j\}_{j\in\mathbf{N}}$ converges uniformly to f in $\partial\mathbf{D}$, and $\partial\mathbf{D}$ is compact, a simple contradiction argument shows that $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ pointwise on $\partial\mathbf{D}$. By Lemma 2.2 (ii), (iii), there exists $\delta > 0$ such that $\min_{\partial\mathbf{D}}|f_j'| \geq \delta$, $\min_{\partial\mathbf{D}}|f'| \geq \delta$. Then a simple computation shows that $\sup_{j\in\mathbf{N}} \|f_j^{(-1)}\|_{C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C})} < \infty$. Since $C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C})$ is compactly imbedded in $C_*^{0,1}(\partial\mathbf{D},\mathbf{C})$, and $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ pointwise on $\partial\mathbf{D}$, a simple contradiction argument shows that $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ in $C_*^{0,1}(\partial\mathbf{D},\mathbf{C})$. Then by Theorem 2.9, case $r = 0$, and by Lemma 2.3 (iii), we have $\lim_{j\to\infty}[f_j'(f_j^{(-1)})]^{-1} = [f'(f^{(-1)})]^{-1}$ in $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$. Since $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ in $C_*^{0,1}(\partial\mathbf{D},\mathbf{C})$, we conclude that $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ in $C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C})$. Now let the statement be true for $m \geq 1$ and $\lim_{j\to\infty} f_j = f$ in $C_*^{m+1,\alpha,0}(\partial\mathbf{D},\mathbf{C})$. By case m we have $\lim_{j\to\infty} f_j^{(-1)} = f^{(-1)}$ in $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$. Then by Lemma 2.3 (iii), by Theorem 2.9, by the limiting relation $\lim_{j\to\infty} f_j' = f'$ in $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, and by equality $[f_j^{(-1)}]' = [f_j'(f_j^{(-1)})]^{-1}$, we conclude that the sequence $\{[f_j^{(-1)}]'\}_{j\in\mathbf{N}}$ converges to $[f^{(-1)}]'$ in $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$, and the proof of case $r = 0$ is complete. We now prove that $\mathbf{J}(C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}) \subseteq C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$. Let $f \in C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$. Then there exists a sequence $\{f_j\}_{j\in\mathbf{N}}$ in $C_*^{\infty}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ such that $\lim_{j\to\infty} f_j = f$ in $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$. Since for j sufficiently large we have $\frac{f_j}{|f_j|} \in C_*^{\infty}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$, then we have $\mathbf{J}\left[\frac{f_j}{|f_j|}\right] \in C_*^{\infty}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}$ for the same j's. Now let $\chi \in C^{\infty}(\mathbf{R}^2,\mathbf{R}^2)$ be such that $\chi(x) = x|x|^{-1}$, for $|x| \geq 1/2$. By [16, Thm. 5.3], and by Lemma 2.1, and by the continuity of the restriction to $\partial\mathbf{D}$ (cf. *e.g.*, [18, Lem. 2.8]), we have

$\lim_{j\to\infty}\frac{f_j}{|f_j|} = \lim_{j\to\infty}\chi(\mathbf{E}[f_j])_{|\partial\mathbf{D}} = \chi(f) = f$ in $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C})$, with $\mathbf{E}$ as in Lemma 2.4. Then the continuity of $\mathbf{J}$ implies that $\mathbf{J}[f]\in C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$. We now consider case $r\geq 1$. We take $R>1$ and we define the operator $\boldsymbol{\Theta}$ of $(C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}})\times C_*^{m,\alpha,0}(\partial\mathbf{D},R\mathbf{D})$ to $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$ by setting $\boldsymbol{\Theta}[f,g] = (\mathbf{E}[f])\circ g - \mathrm{id}_{\partial\mathbf{D}}$, where $\mathrm{id}_{\partial\mathbf{D}}$ denotes the identity map in $\partial\mathbf{D}$. By Theorem 2.9, the operator $\boldsymbol{\Theta}$ is of class C^r. We now observe that for all $f\in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}}$ such that $f(\partial\mathbf{D}) = \partial\mathbf{D}$, we have $\boldsymbol{\Theta}[f,g]=0$ if $g=f^{(-1)}$. We now apply the Implicit Function Theorem to equation $\boldsymbol{\Theta}[f,g]=0$ around the pair $(f_0,f_0^{(-1)})$. By Theorem 2.9, the partial differential of $\boldsymbol{\Theta}$ at $(f_0,f_0^{(-1)})$ with respect to the variable g is defined by $d_g\boldsymbol{\Theta}[f_0,f_0^{(-1)}](h) = f_0'(f_0^{(-1)})h$, for all $h\in C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$. Since f_0' belongs to $C_*^{m+r-1,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, which is contained in $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, and since $f_0'(\tau)\neq 0$ for all $\tau\in\partial\mathbf{D}$, then Lemma 2.3 (ii), (iii) and Theorem 2.9 imply that $d_g\boldsymbol{\Theta}[f_0,f_0^{(-1)}]$ is a linear isomorphism of $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$. Thus the Implicit Function Theorem implies the existence of an open neighborhood $\mathscr{W}_{f_0}$ of f_0 in $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}}$, and of an open neighborhood $\mathscr{V}_{f_0^{(-1)}}$ of $f_0^{(-1)}$ in $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, and of a map $\tilde{\mathbf{J}}_{f_0}$ of class C^r from $\mathscr{W}_{f_0}$ to $\mathscr{V}_{f_0^{(-1)}}$ such that the graph of $\tilde{\mathbf{J}}_{f_0}$ coincides with the set of zeros of $\boldsymbol{\Theta}$ in $\mathscr{W}_{f_0}\times\mathscr{V}_{f_0^{(-1)}}$. By the obvious inclusion of $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C})$ in $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$, and by case $r=0$, $\mathbf{J}$ is continuous on $\mathscr{W}_{f_0}\cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$. Then by possibly shrinking $\mathscr{W}_{f_0}$, we can assume that $\mathbf{J}$ maps $\mathscr{W}_{f_0}\cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$ to $\mathscr{V}_{f_0^{(-1)}}$. Since $\boldsymbol{\Theta}[f,\mathbf{J}[f]]=0$, for all $f\in\mathscr{W}_{f_0}\cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$, we conclude that (2.12) holds. The validity of the formula (2.13) for the first differential follows from formula (2.10) and by the Implicit Function Theorem. □

If $\phi\in C_*^1(\partial\mathbf{D},\mathbf{C})$, and if f is a function of $L=\phi(\partial\mathbf{D})$ to $\mathbf{C}$, then we denote by $\int_\phi f(s)ds$ the line integral of the function f computed with respect to the parametrization $\theta\mapsto\phi(e^{i\theta})$, with $\theta\in[0,2\pi]$, of $\phi(\partial\mathbf{D})$. Let $\phi\in\mathscr{A}_{\partial\mathbf{D}}$. We denote by $\mathrm{ind}[\phi]$ the index of the curve $\theta\mapsto\phi(e^{i\theta})$, $\theta\in[0,2\pi]$ with respect to any of the points of $I[\phi]$. Thus $\mathrm{ind}[\phi]\equiv\frac{1}{2\pi i}\int_\phi\frac{d\xi}{\xi-z}$, for all $z\in I[\phi]$. The map $\mathrm{ind}[\cdot]$ is obviously constantly equal to 1 or to -1 on the open connected components of $\mathscr{A}_{\partial\mathbf{D}}$ in $C_*^1(\partial\mathbf{D},\mathbf{C})$. We set $\mathscr{A}_{\partial\mathbf{D}}^+\equiv\{\phi\in\mathscr{A}_{\partial\mathbf{D}}:\mathrm{ind}[\phi]>0\}$. The following Theorem collects known facts related to singular integrals with Cauchy kernels and to Cauchy type integrals.

Theorem 2.14. *Let* $\alpha\in\,]0,1[$, $m\in\mathbf{N}$, $\phi\in C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}}^+$, $L=\phi(\partial\mathbf{D})$. *Let* $\mathbf{I}$ *be the identity operator in* $C_*^{1,\alpha}(L,\mathbf{C})$. *Then the following statements hold.*
(i) *For all* $f\in C_*^{m,\alpha}(L,\mathbf{C})$, *the singular integral*

$$\mathbf{S}_\phi[f](\tau)\equiv\frac{1}{\pi i}\int_\phi\frac{f(\sigma)}{\sigma-\tau}\,d\sigma,\qquad\forall\tau\in L,\tag{2.15}$$

exists in the sense of the principal value, and $\mathbf{S}_\phi[f](\cdot) \in C_*^{m,\alpha}(L, \mathbf{C})$. *The operator* $\mathbf{S}_\phi$ *defined by* (2.15) *is linear and continuous from* $C_*^{m,\alpha}(L, \mathbf{C})$ *to itself. If* ϕ *coincides with the identity map* $\mathrm{id}_{\partial \mathbf{D}}$, *then we set* $\mathbf{S} \equiv \mathbf{S}_\phi$.

(ii) *For all* $f \in C_*^{m,\alpha}(L, \mathbf{C})$, *the function* $\boldsymbol{\Upsilon}_\phi[f]$ *of* $\mathbf{C} \backslash \{L\}$ *to* $\mathbf{C}$ *defined by*

$$\boldsymbol{\Upsilon}_\phi[f](z) \equiv \frac{1}{2\pi i} \int_\phi \frac{f(\sigma)}{\sigma - z} \, d\sigma, \qquad \forall z \in \mathbf{C} \backslash L,$$

is holomorphic. The function $\boldsymbol{\Upsilon}_\phi[f]_{|I[\phi]}$ *admits a continuous extension to* $\mathrm{cl}\, I[\phi]$, *which we denote by* $\boldsymbol{\Upsilon}_\phi^+[f]$, *and the function* $\boldsymbol{\Upsilon}_\phi[f]_{|E[\phi]}$ *admits a continuous extension to* $\mathrm{cl}\, E[\phi]$, *which we denote by* $\boldsymbol{\Upsilon}_\phi^-[f]$. *Then we have* $\boldsymbol{\Upsilon}_\phi^+[f] \in C^{m,\alpha}(\mathrm{cl}\, I[\phi], \mathbf{C}) \cap H(I[\phi])$, $\boldsymbol{\Upsilon}_\phi^-[f] \in C^0(\mathrm{cl}\, E[\phi], \mathbf{C}) \cap C_*^{m,\alpha}(L, \mathbf{C}) \cap H(E[\phi])$, *and the Plemelj formulas* $\boldsymbol{\Upsilon}_\phi^\pm[f](\tau) = \pm\frac{1}{2} f(\tau) + \frac{1}{2}\mathbf{S}_\phi[f](\tau)$ *for all* $\tau \in L$ *hold. Furthermore,* $\boldsymbol{\Upsilon}_\phi^+[\cdot]$ *defines a linear and continuous operator of* $C_*^{m,\alpha}(L, \mathbf{C})$ *to* $C^{m,\alpha}(\mathrm{cl}\, I[\phi], \mathbf{C})$. *If* ϕ *coincides with the identity map* $\mathrm{id}_{\partial \mathbf{D}}$, *then we set* $\boldsymbol{\Upsilon} \equiv \boldsymbol{\Upsilon}_\phi$.

(iii) *The function* $f \in C_*^{m,\alpha}(L, \mathbf{C})$ *satisfies equation* $(\mathbf{I} - \mathbf{S}_\phi)[f] = 0$, *if and only if there exists a function* $F \in C^{m,\alpha}(\mathrm{cl}\, I[\phi], \mathbf{C}) \cap H(I[\phi])$ *such that* $F(\tau) = f(\tau)$, *for all* $\tau \in L$. *The function F, if it exists, is unique.*

(iv) *The function* $f \in C_*^{m,\alpha}(L, \mathbf{C})$ *satisfies equation* $(\mathbf{I} + \mathbf{S}_\phi)[f] = 0$, *if and only if there exists a function* $F \in C^0(\mathrm{cl}\, E[\phi], \mathbf{C}) \cap C_*^{m,\alpha}(L, \mathbf{C}) \cap H(E[\phi])$ *such that* $\lim_{z \to \infty} F(z) = 0$, *and* $F(\tau) = f(\tau)$, *for all* $\tau \in L$. *The function F, if it exists, is unique.*

(v) *If* $f \in C_*^{1,\alpha}(L, \mathbf{C})$, *then* $(\mathbf{S}_\phi[f])' = \mathbf{S}_\phi[f']$.

For case $m = 0$ of statements (i) and (ii), we refer to Hackbusch [8, Thm. 7.2.5]. Statement (v) follows by Gakhov [6, p. 30]. Case $m > 0$ of statements (i) and (ii) follows by case $m = 0$ and by statement (v). Statements (iii) and (iv) follow by Gakhov [6, p. 27] together with statement (ii).

3. Differentiability properties of the conformal welding operator

As we have said in the introduction, the conformal welding map is a composite function of Riemann maps. Thus we introduce the following Theorem, which summarizes some well-known properties of Riemann maps. For a proof we refer to Ahlfors [1, Ch. 6, §1] together with Pommerenke [26, Thms. 2.6, 3.5, 3.6].

Theorem 3.1. *Let* $m \in \mathbf{N} \backslash \{0\}$, $\alpha \in]0, 1[$. *Let* $\zeta \in C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}$. *Then the following statements hold.*

(i) *There exists a unique homeomorphism* $g[\zeta] \in C^1(\mathbf{C} \backslash \mathbf{D}, \mathbf{C}) \cap H(\mathbf{C} \backslash \mathrm{cl}\, \mathbf{D})$ *of* $\mathbf{C} \backslash \mathbf{D}$ *onto* $\mathrm{cl}\, E[\zeta]$, *such that* $g[\zeta](\infty) \equiv \lim_{z \to \infty} g[\zeta](z) = \infty$, $g[\zeta]'(\infty) \equiv \lim_{z \to \infty} g[\zeta]'(z) \in]0, +\infty[$. *Furthermore* $g[\zeta]_{|\partial \mathbf{D}} \in C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+$.

(ii) *Let* a_1, a_2, a_3 *be three distinct points of* $\partial\mathbf{D}$. *Let* ζ *be orientation preserving. There exists a unique homeomorphism* $f[\zeta] \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C}) \cap H(\mathbf{D})$ *of* $\mathrm{cl}\,\mathbf{D}$ *onto* $\mathrm{cl}\,I[\zeta]$ *such that* $f[\zeta](a_j) = \zeta(a_j)$, *for all* $j = 1, 2, 3$. *Furthermore,* $f[\zeta]_{|\partial\mathbf{D}} \in \mathscr{A}^+_{\partial\mathbf{D}}$.

(iii) *For all* $b \in I[\zeta]$, *there exists a unique homeomorphism* $\tilde{f}[\zeta, b]$ *of* $\mathrm{cl}\,\mathbf{D}$ *onto* $\mathrm{cl}\,I[\zeta]$ *such that* $\tilde{f}[\zeta, b] \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{D}, \mathbf{C}) \cap H(\mathbf{D})$ *and such that* $\tilde{f}[\zeta, b](0) = b$, $\tilde{f}[\zeta, b]'(0) \in]0, +\infty[$. *For short, we set* $\tilde{f}[\zeta] = \tilde{f}[\zeta, 0]$ *if* $b = 0$.

We note that the map $f[\zeta]$ of Theorem 3.1 (ii) depends also on a_1, a_2, a_3. However, throughout the paper, we will assume the three points a_1, a_2, a_3 to be fixed. Thus we have chosen not to display the dependence on a_1, a_2, a_3 in the notation for $f[\zeta]$. By Lemma 2.3 (iv), (v), and by Theorem 3.1, the function $f[g[\zeta]_{|\partial\mathbf{D}}]^{(-1)} \circ g[\zeta]_{|\partial\mathbf{D}}$ belongs to $C^{m,\alpha}_*(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$, for all $\zeta \in C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, with $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0, 1[$. Then we can introduce the following.

Definition 3.2. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0, 1[$. *Let* a_1, a_2, a_3 *be three distinct points of* $\partial\mathbf{D}$. *If* $\zeta \in C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, *then we define as conformal welding map associated to* ζ *(and to the triple* (a_1, a_2, a_3)*), the map*

$$\mathbf{w}[\zeta] \equiv f[g[\zeta]_{|\partial\mathbf{D}}]^{(-1)} \circ g[\zeta]_{|\partial\mathbf{D}}. \tag{3.3}$$

We define as conformal welding operator, the operator $\mathbf{w}[\cdot]$ *of* $C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ *to* $C^{m,\alpha}_*(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$, *which takes* ζ *to* $\mathbf{w}[\zeta]$.

Clearly, one can define the conformal welding map by normalizing $f[\zeta]$ and $g[\zeta]$ in a different way, and the corresponding $\mathbf{w}[\zeta]$ would differ from the $\mathbf{w}[\zeta]$ defined above by a suitable composition with Möbius transformations. Now we introduce the following Theorem, whose first statement has been proved in Lanza and Rogosin [20, Thm. 5.4]. Both statements can be considered as a variant of [17, Thm. 3.10, Thm. 4.7].

Theorem 3.4. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0, 1[$. *Then the following statements hold.*

(i) *The map* $(\zeta, b) \mapsto \tilde{f}[\zeta, b]^{(-1)} \circ \zeta$ *is real analytic from*

$$\mathscr{E}_{m,\alpha} \equiv \{(\zeta, b) \in (C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}) \times \mathbf{C} : b \in I[\zeta]\}$$

to $C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C})$.

(ii) *Let* $r \in \mathbf{N}$. *The map* $(\zeta, b) \mapsto \tilde{f}[\zeta, b]$ *of* $\mathscr{E}_{m+r,\alpha} \cap (C^{m+r,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \times \mathbf{C})$ *to* $C^{m,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C})$ *is of class* C^r.

Proof. Statement (i) is contained in Lanza and Rogosin [20, Thm. 5.4]. By Theorem 3.1, we deduce that $\tilde{f}[\zeta, b] \in C^{\infty}_*(\partial\mathbf{D}, \mathbf{C})$ if $\zeta \in C^{\infty}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, $b \in I[\zeta]$. Then by statement (i), we have $\tilde{f}[\zeta, b]^{(-1)} \circ \zeta \in C^{m,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C})$ if $\zeta \in C^{m,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, $b \in I[\zeta]$. Then statement (ii) follows by statement (i), and by Theorems 2.9, 2.11. □

We now turn to consider the dependence of $f[\zeta]^{(-1)} \circ \zeta$ upon ζ. Since $f[\zeta]$ has been normalized in a different way from $\tilde{f}[\zeta, b]$, we need the following Lemma in order to exploit the previous Theorem.

LEMMA 3.5. *Let a_1, a_2, a_3 be three distinct points of $\partial\mathbf{D}$. Let*

$$\mathbf{A} \equiv \{(z, p_1, p_2, p_3) \in \mathbf{C}^4 : p_1, p_2, p_3 \text{ are distinct}, \\ (z - p_3)(p_1 - p_2)(a_1 - a_3) - (z - p_2)(p_1 - p_3)(a_1 - a_2) \neq 0\}.$$

Let Q be the rational function of $\mathbf{A}$ to $\mathbf{C}$ defined by

$$Q(z, p_1, p_2, p_3) \equiv \frac{a_2(z - p_3)(p_1 - p_2)(a_1 - a_3) - a_3(z - p_2)(p_1 - p_3)(a_1 - a_2)}{(z - p_3)(p_1 - p_2)(a_1 - a_3) - (z - p_2)(p_1 - p_3)(a_1 - a_2)},$$

for all $(z, p_1, p_2, p_3) \in \mathbf{A}$. If p_1, p_2, p_3 are three distinct points of $\partial\mathbf{D}$, and if the triple (p_1, p_2, p_3) induces on $\partial\mathbf{D}$ the same orientation of the triple (a_1, a_2, a_3), then $(z, p_1, p_2, p_3) \in \mathbf{A}$ for all $z \in \mathrm{cl}\,\mathbf{D}$, and $Q(\cdot, p_1, p_2, p_3)$ is the unique homeomorphism of $\mathrm{cl}\,\mathbf{D}$ onto itself which is holomorphic in $\mathbf{D}$ and which maps p_j to a_j, for $j = 1, 2, 3$.

PROOF. By elementary Conformal Mapping Theory (cf. *e.g.*, Ahlfors [1, p. 79]), the function $Q(\cdot, p_1, p_2, p_3)$ is the only linear fractional transformation which maps p_j to a_j. Since p_1, p_2, p_3 are three distinct points of $\partial\mathbf{D}$, the function $Q(\cdot, p_1, p_2, p_3)$ is well-known to map $\partial\mathbf{D}$ onto itself. If the triple (p_1, p_2, p_3) induces on $\partial\mathbf{D}$ the same orientation of (a_1, a_2, a_3), then $Q(\cdot, p_1, p_2, p_3)$ is well-known to be a bijection of $\mathrm{cl}\,\mathbf{D}$ onto itself. The uniqueness follows by the Riemann Mapping Theorem. □

We are now ready to prove the following.

THEOREM 3.6. *Let $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0, 1[$. Let a_1, a_2, a_3 be three distinct points of $\partial\mathbf{D}$. Then the nonlinear operator $\zeta \mapsto f[\zeta]^{(-1)} \circ \zeta$ is real analytic from $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ to $C_*^{m,\alpha}(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, and maps $C_*^{m,\alpha,0}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ to $C_*^{m,\alpha,0}(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$.*

PROOF. Let $\zeta_0 \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, $z_0 \in I[\zeta_0]$. Let $\mathscr{W}$ be an open neighborhood of ζ_0 in $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ such that $z_0 \in I[\zeta]$, for all $\zeta \in \mathscr{W}$. By the uniqueness inferred by the Riemann Mapping Theorem, we have $f[\zeta - z_0] = f[\zeta] - z_0$. Thus there is no loss of generality in assuming that $0 = z_0 \in I[\zeta]$, for all $\zeta \in \mathscr{W}$. By Lemma 3.5, and by the uniqueness inferred by the Riemann Mapping Theorem, we have

$$f[\zeta]^{(-1)} \circ \zeta(\cdot) = Q(\tilde{f}[\zeta]^{(-1)} \circ \zeta(\cdot), \tilde{f}[\zeta]^{(-1)} \circ \zeta(a_1), \tilde{f}[\zeta]^{(-1)} \circ \zeta(a_2), \tilde{f}[\zeta]^{(-1)} \circ \zeta(a_3)),$$

where Q is as in Lemma 3.5. Thus the analyticity of $f[\zeta]^{(-1)} \circ \zeta(\cdot)$ on ζ follows by Theorem 3.4 (i), and by Lemmas 2.3 (ii), (iii) and 3.5. By Theorem 3.1, we have $f[\zeta]^{(-1)} \circ \zeta \in C_*^{\infty}(\partial\mathbf{D}, \partial\mathbf{D})$ if $\zeta \in C_*^{\infty}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^{+}$. Then the last statement follows by the continuity of $\zeta \mapsto f[\zeta]^{(-1)} \circ \zeta$, and by the definition of the spaces $C_*^{m,\alpha,0}$. □

We now show that on a suitable subset $\mathscr{T}_{m,\alpha}$ of $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^{+}$ we have $g[\zeta]_{|\partial\mathbf{D}} = \zeta$, for all $\zeta \in \mathscr{T}_{m,\alpha}$. Thus the conformal welding operator coincides with the nonlinear operator of the previous Theorem on $\mathscr{T}_{m,\alpha}$.

Proposition 3.7. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0,1[$. *Let*

$$D_{m,\alpha} \equiv \Big\{ \zeta \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) : \exists Z \in H(\mathbf{C}\setminus \mathrm{cl}\,\mathbf{D}) \cap C^0(\mathbf{C}\setminus\mathbf{D}, \mathbf{C})$$

$$\text{such that } \zeta = Z_{|\partial\mathbf{D}}, \lim_{z\to\infty} Z'(z) \equiv Z'(\infty) \in \mathbf{R} \Big\}. \tag{3.8}$$

Then the following statements hold.

(i) *If* $\zeta \in D_{m,\alpha}$, *then the map* Z *of* (3.8) *is unique. Furthermore,* $Z'(\infty) = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma$, *and* $\lim_{z\to\infty}\Big\{ Z(z) - \frac{z}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma}\, d\sigma \Big\} = 0$.

(ii) $D_{m,\alpha}$ *is a real Banach subspace of* $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$.

(iii) *Let* $Z[\zeta]$ *be the unique map of statement* (i) *corresponding to* $\zeta \in D_{m,\alpha}$. *Then the set*

$$\mathscr{T}_{m,\alpha} \equiv \{ \zeta \in D_{m,\alpha} \cap \mathscr{A}_{\partial\mathbf{D}}^{+} : Z[\zeta]'(\infty) > 0 \}$$

is open in $D_{m,\alpha}$.

(iv) *If* $\zeta \in \mathscr{T}_{m,\alpha}$, *then* $\lim_{z\to\infty} Z[\zeta](z) = \infty$, *and* $Z[\zeta] = g[\zeta]$.

Proof. We first prove statement (i). If $Z \in H(\mathbf{C}\setminus \mathrm{cl}\,\mathbf{D}) \cap C^0(\mathbf{C}\setminus\mathbf{D}, \mathbf{C})$, then standard properties of holomorphic functions imply that condition $\lim_{z\to\infty} Z'(z) \equiv Z'(\infty) \in \mathbf{C}$ is equivalent to the existence of $a, b \in \mathbf{C}$ and $K \in H(\mathbf{C}\setminus \mathrm{cl}\,\mathbf{D}) \cap C^0(\mathbf{C}\setminus\mathbf{D}, \mathbf{C})$ such that $\lim_{z\to\infty} K(z) = 0$ and $Z(z) = az + b + K(z)$. Moreover, it is easily checked that if such condition holds, then $Z'(\infty) = a = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{Z(\sigma)}{\sigma^2}\, d\sigma$, $b = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{Z(\sigma)}{\sigma}\, d\sigma$. Then by Theorem 2.14 (iv), the membership of ζ in $D_{m,\alpha}$ is equivalent to condition $(\mathbf{I} + \mathbf{S})\Big[\zeta(z) - \frac{z}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma}\, d\sigma\Big] = 0$ together with condition $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma \in \mathbf{R}$. Thus the uniqueness of Z follows. The completeness of $D_{m,\alpha}$ follows by the same argument, by Theorem 2.14 (i), and by the continuous dependence of $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma}\, d\sigma$ and of $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma$ on $\zeta \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$. Statement (iii) follows by Lemma 2.2 (ii) and by the continuous dependence of $Z'(\infty) = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma$ on $\zeta \in D_{m,\alpha}$. We now prove statement (iv). By (i), we have $\lim_{z\to\infty} Z[\zeta](z) = \infty$. Now let $\omega_0 \in I[\zeta]$. Since $Z[\zeta]'(\infty) \neq 0$, then the function $Z[\zeta]$ is injective in a neighborhood of ∞. Since we also have $\zeta \in \mathscr{A}^+_{\partial\mathbf{D}}$ and $\omega_0 \in I[\zeta]$, then a standard argument based on the Argument Principle shows that $Z[\zeta](\cdot) - \omega_0$ does not vanish. Thus the function $[Z[\zeta](1/(\cdot)) - \omega_0]^{-1}$ extends to a holomorphic map in $\mathbf{D}$. Since $\zeta \in \mathscr{A}^+_{\partial\mathbf{D}}$, then the curve $[\zeta(1/(\cdot)) - \omega_0]^{-1}$ is one to one. Then again by the Argument Principle, $[Z[\zeta](1/(\cdot)) - \omega_0]^{-1}$ is one to one in $\mathrm{cl}\,\mathbf{D}$, and a simple topological argument shows that $[Z[\zeta](1/(\cdot)) - \omega_0]^{-1}$ maps $\mathbf{D}$ onto $I[[\zeta(1/(\cdot)) - \omega_0]^{-1}]$. Accordingly, $Z[\zeta]$ is one to one and $Z[\zeta](\mathbf{C}\backslash\mathrm{cl}\,\mathbf{D}) = \mathbf{C}\backslash\mathrm{cl}\,I[\zeta]$. Since $Z[\zeta]'(\infty) > 0$, Theorem 3.1 (i) implies that $Z[\zeta] = g[\zeta]$. □

As an immediate Corollary of Theorem 3.6 and of Proposition 3.7, we obtain the following.

Theorem 3.9. *Let $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$. Then the conformal welding operator is real analytic from the set $\mathscr{T}_{m,\alpha}$ to $C^{m,\alpha}_*(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$.*

We now turn to the differentiability properties of $g[\cdot]$, by means of the following.

Theorem 3.10. *Let $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$, $r \in \mathbf{N}$. Then the map which takes ζ to $g[\zeta]_{|\partial\mathbf{D}}$ is of class C^r in the real sense from $C^{m+r,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ to $C^{m,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$.*

Proof. Let $\zeta_0 \in C^{m+r,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, $z_0 \in I[\zeta_0]$. Let $\mathscr{W}$ be an open neighborhood of ζ_0 in $C^{m+r,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ such that $z_0 \in I[\zeta]$, for all $\zeta \in \mathscr{W}$. By the uniqueness inferred by the Riemann Mapping Theorem, we have $g[\zeta - z_0] = g[\zeta] - z_0$. Thus there is no loss of generality in assuming that $z_0 = 0$, for all $\zeta \in \mathscr{W}$. Let $\tilde{f}[\cdot]$ be as in Theorem 3.1. Clearly, $g[\zeta](z) = [\tilde{f}[1/\zeta](1/z)]^{-1}$, for all $z \in \mathbf{C}\backslash\mathrm{cl}\,\mathbf{D}$, and for all $\zeta \in \mathscr{W}$. In particular, Theorem 3.1 implies that $g[\zeta]_{|\partial\mathbf{D}} \in C^\infty_*(\partial\mathbf{D}, \mathbf{C})$ if $\zeta \in C^\infty_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. Thus we can conclude the proof by Lemma 2.3 (iii), by Theorem 2.9, and by Theorem 3.4 (ii). □

As a consequence of Theorem 3.6, and of Theorem 3.10, we obtain the following.

Theorem 3.11. *Let $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$, $r \in \mathbf{N}$. Then the conformal welding operator maps $C^{m,\alpha}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ to $C^{m,\alpha}_*(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$ and is of class C^r in the real sense from $C^{m+r,\alpha,0}_*(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ to $C^{m,\alpha,0}_*(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$.*

We note that it can be proved that Theorem 3.11 is optimal in the frame of Schauder spaces (cf. [19, Thm. 2.14].)

4. Preliminaries on the conformal sewing problem, and definition of the generalized conformal sewing operator.

The problem of finding a suitable right inverse for the noninjective operator $\mathbf{w}[\cdot]$ is called the conformal sewing problem and is a particular Haseman problem. In this section, by following the classical theory of the Haseman problem we recall a known existence and uniqueness result for the conformal sewing problem. Moreover, we prove a slightly more general regularity result for the solution in order to perform our perturbation analysis of the conformal sewing problem. To introduce a suitable right inverse of $\mathbf{w}[\cdot]$, we first look at the injection which is naturally associated to the conformal welding operator.

Proposition 4.1. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0,1[$. *Let '$\sim$' be the equivalence relation in* $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$ *which is naturally associated to the conformal welding operator, i.e.,* $\zeta_1 \sim \zeta_2$ *if and only if* $\mathbf{w}[\zeta_1] = \mathbf{w}[\zeta_2]$, *with* $\zeta_1, \zeta_2 \in C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. *If* $\zeta_1 \in C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, *then the equivalence class* $[\zeta_1]_\sim$ *of* ζ_1 *with respect to the relation* $\sim$ *contains exactly one element* $\tilde{\zeta}_1 \in \mathscr{T}_{m,\alpha}$ *such that* $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\tilde{\zeta}_1(\sigma)}{\sigma}\, d\sigma = 0$, $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\tilde{\zeta}_1(\sigma)}{\sigma^2}\, d\sigma = 1$. *Furthermore,*

$$\tilde{\zeta}_1 = \left\{\lim_{z\to\infty} g[\zeta_1]'(z)\right\}^{-1}\left\{g[\zeta_1]_{|\partial\mathbf{D}} - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{g[\zeta_1](\sigma)}{\sigma}\, d\sigma\right\}. \tag{4.2}$$

The map $\zeta \mapsto [\zeta]_\sim$ *is a bijection of the set*

$$\tilde{\mathscr{T}}_{m,\alpha} \equiv \left\{\zeta \in \mathscr{T}_{m,\alpha} : \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma}\, d\sigma = 0, \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma^2}\, d\sigma = 1\right\}$$

onto the quotient set $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}/\sim$. *In particular, the conformal welding operator is injective from the set* $\tilde{\mathscr{T}}_{m,\alpha}$ *to* $C_*^{m,\alpha}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$.

Proof. Let $\zeta_1 \in C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. By definition of conformal welding map, we have $\mathbf{w}[\zeta_1] = f[g[\zeta_1]_{|\partial\mathbf{D}}]^{(-1)} \circ g[\zeta_1]_{|\partial\mathbf{D}}$. Also, if we set $\zeta_1^\# \equiv g[\zeta_1]_{|\partial\mathbf{D}}$, we have $g[\zeta_1^\#]_{|\partial\mathbf{D}} = \zeta_1^\#$ and $\zeta_1^\# \in \mathscr{T}_{m,\alpha}$. Then $\mathbf{w}[\zeta_1] = \mathbf{w}[\zeta_1^\#]$. We now note that if $a > 0$, $b \in \mathbf{C}$, then $a\zeta_1^\# + b \in \mathscr{T}_{m,\alpha}$, $g[a\zeta_1^\# + b] = ag[\zeta_1^\#] + b$, $f[a\zeta_1^\# + b] = af[\zeta_1^\#] + b$, and that accordingly $\mathbf{w}[\zeta_1] = \mathbf{w}[a\zeta_1^\# + b]$. In particular, the element $\tilde{\zeta}_1$ defined by the right-hand side of (4.2) belongs to $\mathscr{T}_{m,\alpha}$ and satisfies $\mathbf{w}[\tilde{\zeta}_1] = \mathbf{w}[\zeta_1]$. Since $g[\zeta_1]_{|\partial\mathbf{D}} = \zeta_1^\#$ belongs to $D_{m,\alpha}$, Proposition 3.7 (i) implies that $\lim_{z\to\infty} g[\zeta_1]'(z) = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{g[\zeta_1](\sigma)}{\sigma^2}\, d\sigma$. Then one can easily check that $\tilde{\zeta}_1 \in \tilde{\mathscr{T}}_{m,\alpha}$. Conversely, if $\eta \in \tilde{\mathscr{T}}_{m,\alpha}$, and if $\mathbf{w}[\tilde{\zeta}_1] = \mathbf{w}[\eta]$, then we have $g[\eta]_{|\partial\mathbf{D}} = \eta$, $g[\tilde{\zeta}_1]_{|\partial\mathbf{D}} = \tilde{\zeta}_1$, $f[\tilde{\zeta}_1]^{(-1)} \circ \tilde{\zeta}_1 = f[\eta]^{(-1)} \circ \eta$, and thus $\tilde{\zeta}_1 \circ \eta^{(-1)} = f[\tilde{\zeta}_1] \circ f[\eta]^{(-1)}$ on $\partial\mathbf{D}$. Now the function K of $\mathbf{C}$ to itself defined by $K(z) \equiv g[\tilde{\zeta}_1] \circ g[\eta]^{(-1)}(z)$ if $|z| \geq 1$, $K(z) \equiv f[\tilde{\zeta}_1] \circ f[\eta]^{(-1)}(z)$ if $|z| < 1$, is a homeomorphism of $\mathbf{C}$ to itself, and is holomorphic on $\mathbf{C}\backslash\partial\mathbf{D}$. Then it is well-known that K must be holomorphic on all $\mathbf{C}$. Now the only entire homeomorphisms of $\mathbf{C}$ are the complex affine maps. Thus there exist $a \in \mathbf{C}$, $b \in \mathbf{C}$ such that $K(z) = az + b$, for all $z \in \mathbf{C}$. In particular, $g[\tilde{\zeta}_1] = ag[\eta] + b$, and thus we obtain $\tilde{\zeta}_1 = a\eta + b$. Then by exploiting the assumption that both $\tilde{\zeta}_1$ and η belong to $\tilde{\mathscr{T}}_{m,\alpha}$, one can easily deduce that $a = 1$, $b = 0$. $\square$

The previous Proposition says, in particular, that the natural injection of the quotient set $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}/\sim$ to $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^{+}$ associated to the conformal welding operator can be identified with the restriction of $\mathbf{w}[\cdot]$ to $\tilde{\mathscr{T}}_{m,\alpha}$. Thus we now turn to define a right inverse of the restriction of $\mathbf{w}$ to $\tilde{\mathscr{T}}_{m,\alpha}$. By Proposition 3.7, we easily deduce the following.

Lemma 4.3. *Let* $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$. *Then*

$$\tilde{\mathscr{T}}_{m,\alpha} = \Big\{\zeta \in C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^{+} : \exists Z \in H(\mathbf{C}\backslash \mathrm{cl}\,\mathbf{D}) \cap C^0(\mathbf{C}\backslash\mathbf{D},\mathbf{C}), \text{ such that } \zeta = Z_{|\partial\mathbf{D}}, \lim_{z\to\infty} Z(z) - z = 0\Big\}. \tag{4.4}$$

Also, if $\zeta \in \tilde{\mathscr{T}}_{m,\alpha}$ *and if* Z *is as in* (4.4), *then* $Z = g[\zeta]$.

Proof. If ζ and Z are as in the right hand side of (4.4), then $Z(1/v) - 1/v$ has a removable singularity at $v = 0$ and limiting value 0 at 0. Then by the Cauchy formula, we obtain $\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma}\, d\sigma = 0$. Since $[Z(1/v) - 1/v]'$ must have finite limit at $v = 0$, we deduce that $\lim_{v\to 0} Z'(1/v) = 1$. Then by Proposition 3.7, we have $\zeta \in \tilde{\mathscr{T}}_{m,\alpha}$. Conversely, by Proposition 3.7 we easily deduce that if $\zeta \in \tilde{\mathscr{T}}_{m,\alpha}$, then ζ belongs to the set in the right hand side of (4.4) and $Z = g[\zeta]$. $\square$

As a next step we derive by classical means a system of two integral equations involving ζ, $\mathbf{w}[\zeta]$ (cf. *e.g.*, Lu [22].)

Theorem 4.5. *Let* $\alpha \in]0,1[$. *Let* $\phi \in C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^{+}$. *Let* $F \in C^{0,\alpha}(\mathrm{cl}\, I[\phi],\mathbf{C}) \cap H(I[\phi])$, $G \in C^0(\mathbf{C}\backslash\mathbf{D},\mathbf{C}) \cap H(\mathbf{C}\backslash\mathrm{cl}\,\mathbf{D}) \cap C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$ *such that* $\lim_{z\to\infty} G(z) - z = 0$, *and* $F \circ \phi = G_{|\partial\mathbf{D}}$. *Then* $\zeta \equiv G_{|\partial\mathbf{D}}$ *satisfies the following two equations*

$$\zeta(\tau) + \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma - \tau}\, d\sigma - \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)\phi'(\sigma)}{\phi(\sigma) - \phi(\tau)}\, d\sigma = \tau, \qquad \forall \tau \in \partial\mathbf{D}, \tag{4.6}$$

$$\frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)}{\sigma - \tau}\, d\sigma + \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\zeta(\sigma)\phi'(\sigma)}{\phi(\sigma) - \phi(\tau)}\, d\sigma = \tau, \qquad \forall \tau \in \partial\mathbf{D}. \tag{4.7}$$

In particular, if $m \in \mathbf{N}\backslash\{0\}$, *and* $\zeta \in \tilde{\mathscr{T}}_{m,\alpha}$, *then the pair* $(\phi \equiv \mathbf{w}[\zeta], \zeta)$ *satisfies equations* (4.6) *and* (4.7).

Proof. By Theorem 2.14 (iii) and (iv), we have $\{(\mathbf{I}-\mathbf{S}_{\phi})[\zeta\circ\phi^{(-1)}]\}\circ\phi=0$ and $(\mathbf{I}+\mathbf{S})[\zeta-\mathrm{id}_{\partial\mathbf{D}}]=0$. Then by adding and subtracting such two equations, we obtain (4.6) and (4.7), respectively. □

Next we show, by exploiting known results, that for all $\phi\in C_{*}^{m,\alpha}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}}^{+}$, the system of equations (4.6) and (4.7) has a unique solution. We start by analyzing equation (4.6). To do so, we need some information on the integral operator associated to (4.6). Thus we introduce the following variant of a result of Kantorovich and Akilov [12, Thm. 4 p. 363, Rmk. 2 p. 365] (see also Gorenflo and Vessella [7, Thm. 4.1.7, p. 69].)

Lemma 4.8. *Let $\alpha\in]0,1[$. Let $k(\cdot,\cdot)$ be a complex valued continuous function on the set $\mathbf{B}\equiv\{(\tau,\xi)\in(\partial\mathbf{D})^{2}:\tau\neq\xi\}$. Let*

$$M_{1}\equiv\sup_{(\tau,\xi)\in\mathbf{B}}|k(\tau,\xi)|\,|\xi-\tau|^{1-\alpha}<+\infty. \tag{4.9}$$

Then the following statements hold.

(i) *The integral*

$$\mathbf{U}[\gamma](\tau)\equiv\int_{\partial\mathbf{D}}k(\tau,\xi)\gamma(\xi)d\xi$$

is convergent for all $\tau\in\partial\mathbf{D}$ and for all $\gamma\in L^{\infty}(\partial\mathbf{D},\mathbf{C})$.

(ii) *Let $\mathscr{X}$ be normed space continuously imbedded in $L^{\infty}(\partial\mathbf{D},\mathbf{C})$. Let $\rho>0$. If*

$$M_{2}\equiv\sup\left\{\left|\int_{\partial\mathbf{D}\setminus\mathbf{L}(\tau_{1},\tau_{2})}(k(\tau_{2},\xi)-k(\tau_{1},\xi))\gamma(\xi)d\xi\right|\|\gamma\|_{\mathscr{X}}^{-1}|\tau_{2}-\tau_{1}|^{-\alpha}\right.$$
$$\left.\tau_{1},\tau_{2}\in\partial\mathbf{D},0<|\tau_{2}-\tau_{1}|\leq\rho,\gamma\in\mathscr{X}\setminus\{0\}\right\}<+\infty, \tag{4.10}$$

where $\mathbf{L}(\tau_{1},\tau_{2})\equiv\{\xi\in\partial\mathbf{D}:|\xi-\tau_{1}|<2|\tau_{2}-\tau_{1}|\}$. Then $\mathbf{U}$ defines a linear and continuous operator of $\mathscr{X}$ to $C_{*}^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$.

(iii) *If $\partial_{\tau}k(\cdot,\cdot)$ exists and is continuous, and if*

$$M_{3}\equiv\sup_{(\tau,\xi)\in\mathbf{B}}|\partial_{\tau}k(\tau,\xi)|\,|\xi-\tau|^{2-\alpha}<+\infty, \tag{4.11}$$

then $\mathbf{U}$ defines a linear and continuous operator of $L^{\infty}(\partial\mathbf{D},\mathbf{C})$ to $C_{*}^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$.

Proof. Let $|d\xi|$ denote the usual arc-length measure on $\partial\mathbf{D}$. It is well-known and easy to verify that $c_{\alpha}\equiv\sup_{\tau\in\partial\mathbf{D}}\int_{\partial\mathbf{D}}\frac{|d\xi|}{|\tau-\xi|^{1-\alpha}}<\infty$. Then by assumption (4.9), we have

$$|\mathbf{U}[\gamma](\tau)|\leq M_{1}c_{\alpha}\|\gamma\|_{L^{\infty}(\partial\mathbf{D},\mathbf{C})},\qquad\forall\gamma\in L^{\infty}(\partial\mathbf{D},\mathbf{C}),$$

and for all $\tau \in \partial\mathbf{D}$, and statement (i) holds. We now prove statement (ii). We can clearly assume $\rho \leq 2^{-1}$. Let τ_1, τ_2 be two points of $\partial\mathbf{D}$. If $|\tau_1 - \tau_2| \geq \rho$, then we have

$$|\mathbf{U}[\gamma](\tau_2) - \mathbf{U}[\gamma](\tau_1)|\,|\tau_2 - \tau_1|^{-\alpha} \leq 2\rho^{-\alpha} \sup_{\tau \in \partial\mathbf{D}} |\mathbf{U}[\gamma](\tau)| \leq 2\rho^{-\alpha} c_\alpha M_1 \|\gamma\|_{L^\infty(\partial\mathbf{D},\mathbf{C})}. \tag{4.12}$$

We now assume that $0 < |\tau_1 - \tau_2| \leq \rho$. Then we have

$$\begin{aligned} |\mathbf{U}[\gamma](\tau_2) - \mathbf{U}[\gamma](\tau_1)| &\leq M_2 \|\gamma\|_{\mathscr{X}} |\tau_1 - \tau_2|^\alpha \\ &\quad + \left\{ \int_{\mathbf{L}(\tau_1,\tau_2)} |k(\tau_2,\xi)|\,|d\xi| + \int_{\mathbf{L}(\tau_1,\tau_2)} |k(\tau_1,\xi)|\,|d\xi| \right\} \|\gamma\|_{L^\infty(\partial\mathbf{D},\mathbf{C})}. \end{aligned} \tag{4.13}$$

Now it can be readily verified that there exists $c'_\alpha > 0$ independent of τ_1, τ_2 such that

$$\int_{\mathbf{L}(\tau_1,\tau_2)} \frac{|d\xi|}{|\tau_2 - \xi|^{1-\alpha}} \leq c'_\alpha |\tau_2 - \tau_1|^\alpha, \qquad \int_{\mathbf{L}(\tau_1,\tau_2)} \frac{|d\xi|}{|\tau_1 - \xi|^{1-\alpha}} \leq c'_\alpha |\tau_2 - \tau_1|^\alpha. \tag{4.14}$$

By assumption (4.9), by inequalities (4.12)–(4.14), we conclude that there exists $c'' > 0$ depending only on α, c_α, c'_α, ρ, M_1, M_2, and on the norm of the imbedding of $\mathscr{X}$ into $L^\infty(\partial\mathbf{D},\mathbf{C})$ such that

$$|\mathbf{U}[\gamma](\tau_2) - \mathbf{U}[\gamma](\tau_1)|\,|\tau_2 - \tau_1|^{-\alpha} \leq c'' \|\gamma\|_{\mathscr{X}},$$

for all $\gamma \in \mathscr{X}$, for all $\tau_1, \tau_2 \in \partial\mathbf{D}$. Then statement (ii) follows. We now prove statement (iii). Let $0 < \rho < 2^{-1}$. Let $0 < |\tau_1 - \tau_2| \leq \rho$. By parametrizing the arc joining τ_1 to τ_2 and contained in $\mathbf{L}(\tau_1,\tau_2)$ by the map $\theta \mapsto e^{i\theta}\tau_1$, and by applying the Mean Value Inequality, we obtain that

$$\begin{aligned} |k(\tau_2,\xi) - k(\tau_1,\xi)| &\leq \frac{\pi}{2}|\tau_2 - \tau_1| \sup_{\substack{|\eta-\tau_1| \leq |\tau_2-\tau_1| \\ \eta \in \partial\mathbf{D}}} |\partial_\tau k(\eta,\xi)| \\ &\leq \frac{\pi}{2}|\tau_2 - \tau_1| \sup_{\substack{|\eta-\tau_1| \leq |\tau_2-\tau_1| \\ \eta \in \partial\mathbf{D}}} \left(\frac{M_3}{|\eta - \xi|^{2-\alpha}} \right), \end{aligned}$$

for all $\xi \in \partial\mathbf{D} \backslash \mathbf{L}(\tau_1,\tau_2)$. Now, if $\xi \in \partial\mathbf{D} \backslash \mathbf{L}(\tau_1,\tau_2)$, and if $\eta \in \partial\mathbf{D}$ satisfies inequality $|\eta - \tau_1| \leq |\tau_2 - \tau_1|$, then we have $|\eta - \xi| \geq |\tau_1 - \xi| - |\eta - \tau_1| \geq |\tau_1 - \xi| - |\tau_2 - \tau_1| \geq 2^{-1}|\tau_1 - \xi|$. Furthermore, it is well-known and easy to verify that there exists $c'''_\alpha > 0$ independent of τ_1, τ_2 such that

$$\int_{\partial\mathbf{D} \backslash \mathbf{L}(\tau_1,\tau_2)} |\tau_1 - \xi|^{\alpha-2} |d\xi| \leq c'''_\alpha |\tau_2 - \tau_1|^{\alpha-1}.$$

Then the following inequality holds

$$\int_{\partial\mathbf{D}\setminus\mathbf{L}(\tau_1,\tau_2)} |k(\tau_2,\xi)-k(\tau_1,\xi)|\,|d\xi| \le c_\alpha'''\pi 2^{1-\alpha}M_3|\tau_2-\tau_1|^\alpha, \tag{4.15}$$

and thus the supremum in (4.10) is finite and the conclusion follows by statement (ii) and by taking $\mathscr{X} = L^\infty(\partial\mathbf{D},\mathbf{C})$. □

By the previous Lemma, and by arguing as in Lu [22, p. 418], we obtain the following.

Proposition 4.16. *Let $\alpha,\beta \in]0,1[$. Let $\phi \in C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$. If $\gamma \in C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$, then the map $\mathbf{U}_\phi[\gamma](\cdot)$ defined by*

$$\mathbf{U}_\phi[\gamma](\tau) \equiv \frac{1}{2\pi i}\int_{\partial\mathbf{D}}\left(\frac{1}{\sigma-\tau} - \frac{\phi'(\sigma)}{\phi(\sigma)-\phi(\tau)}\right)\gamma(\sigma)d\sigma, \qquad \forall\tau\in\partial\mathbf{D}, \tag{4.17}$$

belongs to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$. The operator $\mathbf{U}_\phi[\cdot]$ of $C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$ defined by (4.17) *is compact.*

Proof. By arguing as in Lu [22, p. 418], we can show that there exists a constant $c_\phi > 0$ depending only on ϕ such that

$$\left|\frac{\phi(\sigma)-\phi(\tau)}{\sigma-\tau} - \phi'(\sigma)\right| \le c_\phi|\sigma-\tau|^\alpha, \tag{4.18}$$

for all $\tau,\sigma\in\partial\mathbf{D}$ with $\tau\neq\sigma$. Then we obtain that

$$\left|\frac{1}{\sigma-\tau} - \frac{\phi'(\sigma)}{\phi(\sigma)-\phi(\tau)}\right| = \frac{1}{|\phi(\sigma)-\phi(\tau)|}\left|\frac{\phi(\sigma)-\phi(\tau)}{\sigma-\tau} - \phi'(\sigma)\right| \le \frac{c_\phi}{l_{\partial\mathbf{D}}[\phi]}|\sigma-\tau|^{\alpha-1}.$$

Thus the kernel $k(\tau,\sigma) \equiv \frac{1}{\sigma-\tau} - \frac{\phi'(\sigma)}{\phi(\sigma)-\phi(\tau)}$ satisfies inequality (4.9). Similarly, one can prove that k satisfies inequality (4.11). Thus Lemma 4.8 (iii) implies that $\mathbf{U}_\phi$ maps $L^\infty(\partial\mathbf{D},\mathbf{C})$ to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$ with continuity. Since $C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$ is well-known to be compactly imbedded in $L^\infty(\partial\mathbf{D},\mathbf{C})$, the proof is complete. □

We now introduce the following two statements, which we exploit later. For the sake of completeness we include a proof.

Proposition 4.19. *Let $\alpha\in]0,1[$. Let $\phi_1,\phi_2 \in C_*^{1,\alpha}(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}_{\partial\mathbf{D}}^+$, $L_1 \equiv \phi_1(\partial\mathbf{D})$, $L_2\equiv\phi_2(\partial\mathbf{D})$. Let $F\in C^0(\mathrm{cl}\, I[\phi_1],\mathbf{C})\cap H(I[\phi_1])$, $G\in C^0(\mathbf{C}\setminus I[\phi_2],\mathbf{C})\cap H(\mathbf{C}\setminus\mathrm{cl}\, I[\phi_2])$. Let $F_{|\phi_1(\partial\mathbf{D})}\in C_*^{0,\alpha}(L_1,\mathbf{C})$, $G_{|\phi_2(\partial\mathbf{D})}\in C_*^{0,\alpha}(L_2,\mathbf{C})$, $\lim_{z\to\infty} G(z) = 0$, $F(\phi_1(\tau)) = G(\phi_2(\tau))$, for all $\tau\in\partial\mathbf{D}$. Then both F and G vanish identically.*

Proof. Clearly, $F^n(\phi_1(\tau)) = G^n(\phi_2(\tau))$, for all $\tau\in\partial\mathbf{D}$, $n\in\mathbf{N}\setminus\{0\}$. By Theorem 2.14, we have $\{(\mathbf{I}+\mathbf{S}_{\phi_2})[G^n_{|\phi_2(\partial\mathbf{D})}]\}\circ\phi_2 = 0$, $\{(\mathbf{I}-\mathbf{S}_{\phi_1})[G^n\circ\phi_2\circ\phi_1^{(-1)}]\}\circ\phi_1 = 0$. Then by adding such two equalities, we obtain $(\mathbf{I}+\mathbf{U}_{\phi_1}-\mathbf{U}_{\phi_2})[G^n\circ\phi_2] = 0$. Since $\mathbf{U}_{\phi_1}-\mathbf{U}_{\phi_2}$ is compact in $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$, the kernel of

$\mathbf{I}+\mathbf{U}_{\phi_1}-\mathbf{U}_{\phi_2}$ is finite dimensional. Thus the family $\{G^n \circ \phi_2\}_{n\in\mathbf{N}}$ must be linearly dependent, and accordingly there exists a natural k such that $G^0\circ\phi_2,\dots,$ $G^k\circ\phi_2$ are linearly dependent. If G is not identically zero, then the function $G\circ\phi_2$ cannot be constant on $\partial\mathbf{D}$. Thus there exist $k+1$ distinct points $\xi_0,\dots,\xi_k$ in $G\circ\phi_2(\partial\mathbf{D})$. By the linear dependence of $G^0\circ\phi_2,\dots,G^k\circ\phi_2$, we deduce that the determinant of the Vandermonde matrix of the numbers $\xi_0,\dots,\xi_k$ must vanish, contrary to the assumption that $\xi_0,\dots,\xi_k$ be distinct. Thus $G=0$, and accordingly F is identically zero. □

Then we have the following Theorem (cf. Lu [22, pp. 419–420].)

Theorem 4.20. *Let $\alpha\in]0,1[$, $\phi\in C^{1,\alpha}_*(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}^+_{\partial\mathbf{D}}$. Let $\mathbf{I}$ be the identity operator of $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$. Let $\mathbf{U}_\phi$ be the operator of $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ to itself defined in* (4.17). *Then $\mathbf{I}+\mathbf{U}_\phi$ is a complex linear homeomorphism of $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ to itself.*

Proof. By Proposition 4.16, the operator $\mathbf{I}+\mathbf{U}_\phi$ is a compact perturbation of the identity. Thus by the Fredholm Alternative, and by the Open Mapping Theorem, it suffices to show that $\mathbf{I}+\mathbf{U}_\phi$ is injective. If $\gamma\in C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ and $(\mathbf{I}+\mathbf{U}_\phi)[\gamma]=0$, then we have $\frac{1}{2}(\mathbf{I}+\mathbf{S})[\gamma]+\frac{1}{2}\{(\mathbf{I}-\mathbf{S}_\phi)\cdot[\gamma\circ\phi^{(-1)}]\}\circ\phi=0$, and by Theorem 2.14 (ii) we have $\boldsymbol{\Upsilon}^+[\gamma]=\boldsymbol{\Upsilon}^-_\phi[\gamma\circ\phi^{(-1)}]\circ\phi$ on $\partial\mathbf{D}$. Then by Proposition 4.19, it follows that $\boldsymbol{\Upsilon}^+[\gamma]=0$, $\boldsymbol{\Upsilon}^-_\phi[\gamma\circ\phi^{(-1)}]=0$, and accordingly, $(\mathbf{I}+\mathbf{S})[\gamma]=0$, $\{(\mathbf{I}-\mathbf{S}_\phi)[\gamma\circ\phi^{(-1)}]\}\circ\phi=0$ by Plemelj's formula. By Theorem 2.14 (iii), (iv), there exist $F\in C^0(\operatorname{cl} I[\phi])\cap H(I[\phi])$, $G\in C^0(\mathbf{C}\backslash\mathbf{D},\mathbf{C})\cap H(\mathbf{C}\backslash\operatorname{cl}\mathbf{D})$ such that $\lim_{z\to\infty}G(z)=0$, $\gamma=G_{|\partial\mathbf{D}}$, $\gamma\circ\phi^{(-1)}=F_{|\phi(\partial\mathbf{D})}$, and thus $G=F\circ\phi$ on $\partial\mathbf{D}$. Then by Lemma 2.3 (i), (iv), (v) and by Proposition 4.19, we conclude that F and G are identically zero. Thus $\gamma=0$. □

The previous Theorem shows that equation $\mathbf{U}_\phi[\gamma]+\gamma=f$ can be uniquely solved in $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ if f is in $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$. We now wish to prove that the solution is of class $C^{m,\alpha}_*$ if f is of class $C^{m,\alpha}_*$. To do so, we prove the following technical Lemma.

Lemma 4.21. *Let $m\in\mathbf{N}\backslash\{0\}$, $0<\beta<\alpha<1$. Let $\phi\in C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C})\cap\mathscr{A}^+_{\partial\mathbf{D}}$. Then the operator $\mathbf{U}_\phi$ defined in* (4.17) *maps $C^{m,\beta}_*(\partial\mathbf{D},\mathbf{C})$ to $C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ with continuity.*

Proof. By well-known properties of the Cauchy integral, and by Lemma 2.3 and Theorem 2.14, the operator $\mathbf{U}_\phi$ maps $C^{m,\beta}_*(\partial\mathbf{D},\mathbf{C})$ to itself with continuity. Since $C^{m,\beta}_*(\partial\mathbf{D},\mathbf{C})$ is imbedded with continuity in $C^{m}_*(\partial\mathbf{D},\mathbf{C})$, it suffices to show that the operator $\gamma\mapsto(\mathbf{U}_\phi[\gamma])^{(m)}(\cdot)$ maps $C^{m,\beta}_*(\partial\mathbf{D},\mathbf{C})$ to $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ with continuity. To do so, we find convenient to introduce the following notation. We set

$$\tilde{\mathbf{S}}[\phi,\gamma] \equiv \mathbf{S}_\phi[\gamma \circ \phi^{(-1)}] \circ \phi \tag{4.22}$$

for all $\gamma \in C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$. As a first step, we show by induction that for $h = 0,\dots,m$ there exists a continuous operator $\mathbf{V}_{\phi,h}$ of $C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{m-h,\alpha}(\partial\mathbf{D},\mathbf{C})$ such that

$$2\frac{d^h}{d\tau^h}\mathbf{U}_\phi[\gamma] = \mathbf{S}[\gamma^{(h)}] - \tilde{\mathbf{S}}\left[\phi,\frac{\gamma^{(h)}}{(\phi')^h}\right](\phi')^h + \mathbf{V}_{\phi,h}[\gamma]. \tag{4.23}$$

Case $h = 0$ holds with $\mathbf{V}_{\phi,h} = 0$. We now assume that $\mathbf{V}_{\phi,h}$ exists for $0 \le h < m$, and prove the existence of $\mathbf{V}_{\phi,h+1}$. By Theorem 2.14 (v), we have

$$\frac{d}{d\tau}\{\tilde{\mathbf{S}}[\phi,\gamma](\tau)\} = \tilde{\mathbf{S}}\left[\phi,\frac{\gamma'}{\phi'}\right](\tau)\phi'(\tau), \qquad \forall\tau \in \partial\mathbf{D},$$

for all $\gamma \in C_*^{1,\beta}(\partial\mathbf{D},\mathbf{C})$. Then if $h = 0$, we can take $\mathbf{V}_{\phi,h+1} = 0$. If $h \ge 1$, then $m \ge 2$ and by differentiating formula (4.23), we can easily see that the same formula (4.23) holds for $h+1$ provided that

$$\begin{aligned}\mathbf{V}_{\phi,h+1}[\gamma](\cdot) \equiv{}& h\tilde{\mathbf{S}}\left[\phi,\frac{\gamma^{(h)}\phi^{(2)}}{(\phi')^{h+2}}\right](\cdot)(\phi'(\cdot))^{h+1} - h\tilde{\mathbf{S}}\left[\phi,\frac{\gamma^{(h)}}{(\phi')^h}\right](\cdot)(\phi'(\cdot))^{h-1}\phi^{(2)}(\cdot)\\ &+ (\mathbf{V}_{\phi,h}[\gamma])'(\cdot).\end{aligned}$$

By Lemma 2.3 (ii), (iii), by Theorem 2.14 (i), by the continuity of the imbeddings of $C_*^{m-h,\beta}(\partial\mathbf{D},\mathbf{C})$, of $C_*^{m-1,\alpha}(\partial\mathbf{D},\mathbf{C})$, and of $C_*^{m-2,\alpha}(\partial\mathbf{D},\mathbf{C})$ in $C_*^{m-h-1,\alpha}(\partial\mathbf{D},\mathbf{C})$, and by the continuity of $\mathbf{V}_{\phi,h}[\cdot]$ from $C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{m-h,\alpha}(\partial\mathbf{D},\mathbf{C})$, we deduce that $\mathbf{V}_{\phi,h+1}[\cdot]$ is continuous from $C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$ to the space $C_*^{m-(h+1),\alpha}(\partial\mathbf{D},\mathbf{C})$. Thus we conclude that formula (4.23) holds for all $h = 0,\dots,m$, and that in particular it holds for $h = m$. To conclude the proof it suffices to show that the operator which takes $\gamma \in C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$ to $\mathbf{S}[\gamma^{(m)}] - \tilde{\mathbf{S}}\left[\phi,\frac{\gamma^{(m)}}{(\phi')^m}\right](\phi')^m$ is linear and continuous from $C_*^{m,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$. Since $\mathbf{U}_\phi$ maps continuously $C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$, then it suffices to show that the integral operator which takes $\gamma(\cdot)$ to $\int_{\partial\mathbf{D}} k_m(\cdot,\xi)\gamma(\xi)d\xi$, with

$$k_m(\tau,\xi) \equiv \frac{\phi'(\xi)}{\phi(\xi)-\phi(\tau)}\left\{\frac{(\phi'(\tau))^m - (\phi'(\xi))^m}{(\phi'(\xi))^m}\right\}$$

maps continuously $C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$ to $C_*^{0,\alpha}(\partial\mathbf{D},\mathbf{C})$. We shall exploit Lemma 4.8. A simple computation based on the Hölder continuity of ϕ', on assumption $l_{\partial\mathbf{D}}[\phi] > 0$, and on Lemma 2.2 (i), (iii) shows that $k_m(\cdot,\cdot)$ satisfies (4.9). We now consider case $m = 1$ and we show that we can apply statement (ii) of Lemma 4.8 with $\mathscr{X} = C_*^{0,\beta}(\partial\mathbf{D},\mathbf{C})$. Clearly,

$$\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}(k_1(\tau_2,\xi)-k_1(\tau_1,\xi))\gamma(\xi)d\xi\right|$$
$$\leq\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{(\phi'(\tau_2)-\phi'(\xi))(\phi(\tau_2)-\phi(\tau_1))}{(\phi(\xi)-\phi(\tau_2))(\phi(\xi)-\phi(\tau_1))}\gamma(\xi)d\xi\right|$$
$$+|\phi'(\tau_2)-\phi'(\tau_1)|\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\gamma(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi\right|$$
$$\leq\frac{\|\gamma\|_{L^\infty(\partial\mathbf{D},\mathbf{C})}}{l_{\partial\mathbf{D}}[\phi]^2}\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{|\phi':\partial\mathbf{D}|_\alpha|\tau_2-\xi|^\alpha|\phi(\tau_2)-\phi(\tau_1)|}{|\xi-\tau_2|\,|\xi-\tau_1|}|d\xi|$$
$$+|\phi':\partial\mathbf{D}|_\alpha|\tau_2-\tau_1|^\alpha\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\gamma(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi\right|, \tag{4.24}$$

and

$$\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\gamma(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi=\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\left(\frac{\gamma(\xi)}{\phi'(\xi)}-\frac{\gamma(\tau_1)}{\phi'(\tau_1)}\right)\frac{\phi'(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi$$
$$+\frac{\gamma(\tau_1)}{\phi'(\tau_1)}\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\phi'(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi.$$

Now, it can be easily verified that $\frac{2}{3}|\xi-\tau_2|\leq|\xi-\tau_1|\leq 2|\xi-\tau_2|$ for all $\tau_1,\tau_2\in\partial\mathbf{D}$, and $\xi\in\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)$ and that $\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{|\tau_1-\tau_2|^{1-\alpha}}{|\xi-\tau_1|^{2-\alpha}}|d\xi|$ is bounded uniformly in $\tau_1,\tau_2\in\partial\mathbf{D}$. Then by the Lipschitz continuity of ϕ (see Lemma 2.3 (i)), there exists $c>0$ depending only on α and ϕ such that the right hand side of inequality (4.24) is less or equal to

$$c\Biggl\{\|\gamma\|_{L^\infty(\partial\mathbf{D},\mathbf{C})}+\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\left(\frac{\gamma(\xi)}{\phi'(\xi)}-\frac{\gamma(\tau_1)}{\phi'(\tau_1)}\right)\frac{\phi'(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi\right|$$
$$+\left|\frac{\gamma(\tau_1)}{\phi'(\tau_1)}\right|\left|\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\phi'(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi\right|\Biggr\}|\tau_2-\tau_1|^\alpha. \tag{4.25}$$

The membership of γ in $C^{0,\beta}_*(\partial\mathbf{D},\mathbf{C})$, of $\frac{1}{\phi'(\cdot)}$ in $C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$, Lemma 2.2 (i), Lemma 2.3 (i), (ii), and the finiteness of $\sup_{\tau\in\partial\mathbf{D}}\int_{\partial\mathbf{D}}\frac{|d\xi|}{|\xi-\tau|^{1-\beta}}$ imply that there exists $c'>0$ depending only on α and ϕ such that the first integral in (4.25) is bounded by $c'\|\gamma\|_{C^{0,\beta}_*(\partial\mathbf{D},\mathbf{C})}$ uniformly in $\tau_1,\tau_2\in\partial\mathbf{D}$. By Lemma 2.2 (iii) we have $|\gamma(\tau_1)/\phi'(\tau_1)|\leq l_{\partial\mathbf{D}}[\phi]^{-1}\|\gamma\|_{L^\infty(\partial\mathbf{D},\mathbf{C})}$. Thus it suffices to show that $I(\tau_1,\tau_2)\equiv\int_{\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2)}\frac{\phi'(\xi)}{(\phi(\xi)-\phi(\tau_1))}d\xi$ is bounded for $0<|\tau_1-\tau_2|<2^{-1}$. Let

$\partial\mathbf{D}\backslash\mathbf{L}(\tau_1,\tau_2) = \{e^{i\theta} : \theta_a \le \theta \le \theta_b\}$ with $\tau_1 = e^{i\theta_1}$, $\theta_1 < \theta_a < \theta_b < \theta_1 + 2\pi$, $a \equiv e^{i\theta_a}$, $b \equiv e^{i\theta_b}$. By a well-known computation, we have

$$\exp\{I(\tau_1,\tau_2)\} = \frac{\phi(b)-\phi(\tau_1)}{\phi(a)-\phi(\tau_1)}. \tag{4.26}$$

Since $\phi \in C^{1,\alpha}_*(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}$, a simple computation based on Lemma 2.2 (i), and on inequality (4.18) shows that there exists $\delta \in]0, 2^{-1}[$ such that $\left|\frac{\phi(b)-\phi(\tau_1)}{\phi(a)-\phi(\tau_1)} + 1\right| < \frac{1}{4}$ whenever $0 < |\tau_1 - \tau_2| < \delta$. Since $I(\cdot,\cdot)$ is continuous on $\mathbf{B} \equiv \{(\tau,\xi) \in (\partial\mathbf{D})^2 : \tau \ne \xi\}$, we have $\sup\{|I(\tau_1,\tau_2)| : \delta < |\tau_1 - \tau_2| \le 2^{-1}\} < +\infty$. Thus it suffices to show that $\sup\{|I(\tau_1,\tau_2)| : 0 < |\tau_1 - \tau_2| \le \delta\} < +\infty$. Since the map I is continuous on $\mathbf{B}_\delta \equiv \{(\tau_1,\tau_2) \in \mathbf{B} : |\tau_1 - \tau_2| \le \delta\}$ and $\mathbf{B}_\delta$ has a finite number of connected components, then the set $I(\mathbf{B}_\delta)$ has at most a finite number of connected components. Now the set $\mathbf{A} \equiv \{w \in \mathbf{C} : |\exp(w) + 1| < 4^{-1}\}$ is a countable union of bounded disjoint connected sets. Since $I(\mathbf{B}_\delta) \subseteq \mathbf{A}$, then the set $I(\mathbf{B}_\delta)$ must be bounded. Then by Lemma 4.8 (ii), the proof of case $m = 1$ is complete. We now consider case $m \ge 2$. A simple computation based on the Hölder continuity of ϕ', on assumption $l_{\partial\mathbf{D}}[\phi] > 0$, and on Lemma 2.2 (i), (iii), shows that $k_m(\cdot,\cdot)$ satisfies the assumptions of Lemma 4.8 (iii), and thus the proof of case $m \ge 2$ is complete. □

By the previous statement, by the compactness of the imbedding of $C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ to $C^{m,\beta}_*(\partial\mathbf{D},\mathbf{C})$ for $0 < \beta < \alpha$, by the injectivity of $\mathbf{I} + \mathbf{U}_\phi$ inferred by Theorem 4.20, by the Fredholm Alternative, and by the Open Mapping Theorem, we immediately deduce the following.

THEOREM 4.27. *Let $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$. Let $\phi \in C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$. Then $\mathbf{I} + \mathbf{U}_\phi$ is a complex linear homeomorphism of $C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ to itself.*

We now turn to the analysis of the system of equations (4.6)–(4.7), and we prove the following.

THEOREM 4.28. *Let $m \in \mathbf{N}\backslash\{0\}$, $\alpha \in]0,1[$. Let a_1, a_2, a_3 be three distinct points of $\partial\mathbf{D}$. Then the following statements hold.*

(i) *If $\phi \in C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$, then the system of equations* (4.6) *and* (4.7) *admits a unique solution $\zeta \in C^{m,\alpha}_*(\partial\mathbf{D},\mathbf{C})$, which we denote by $\mathbf{s}[\phi]$ and define as the generalized conformal sewing corresponding to ϕ. The system of equations* (4.6) *and* (4.7) *is equivalent to equation* (4.6).

(ii) *If $\phi \in C^{1,\alpha}_*(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}^+_{\partial\mathbf{D}}$, then there exist two uniquely determined functions $G \in C^0(\mathbf{C}\backslash\mathbf{D},\mathbf{C}) \cap H(\mathbf{C}\backslash\mathrm{cl}\,\mathbf{D}) \cap C^{0,\alpha}_*(\partial\mathbf{D},\mathbf{C})$ and $F \in C^{0,\alpha}(\mathrm{cl}\, I[\phi],\mathbf{C}) \cap H(I[\phi])$ such that $\lim_{z\to\infty} G(z) - z = 0$, and $F \circ \phi = G_{|\partial\mathbf{D}}$. Moreover, $G_{|\partial\mathbf{D}} = \mathbf{s}[\phi]$, and $F_{|\phi(\partial\mathbf{D})} = \mathbf{s}[\phi] \circ \phi^{(-1)}$. We denote such unique functions F and G by $F[\phi]$ and $G[\phi]$, respectively.*

(iii) *If* $\phi \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, *then the function* $\mathbf{s}[\phi]$ *belongs to* $\tilde{\mathscr{T}}_{m,\alpha}$, *and* $G[\phi] = g[\mathbf{s}[\phi]]$, *and* $F[\phi] = f[\mathbf{s}[\phi]] \circ (f[\phi])^{(-1)} \in C^{m,\alpha}(\mathrm{cl}\, I[\phi], \mathbf{C}) \cap H(I[\phi])$. *Furthermore,* $\mathbf{s}[f[\phi]^{(-1)} \circ \phi] = \mathbf{s}[\phi]$.

(iv) *If* $\phi \in C_*^{m,\alpha}(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, *and if* $\phi(a_j) = a_j$ *for* $j = 1, 2, 3$, *then* $f[\mathbf{s}[\phi]] = F[\phi]$, *and* $\mathbf{w}[\mathbf{s}[\phi]] = \phi$.

Proof. We first prove statement (i). By Theorem 4.27, equation (4.6) admits a unique solution $\zeta \in C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C})$. We now show that such solution satisfies also equation (4.7). We note that by Theorem 2.14 (ii), equation (4.6) can be rewritten as

$$\Upsilon^+[\zeta - \mathrm{id}_{\partial\mathbf{D}}](\tau) = \Upsilon_\phi^-[\zeta \circ \phi^{(-1)}] \circ \phi(\tau), \qquad \forall \tau \in \partial\mathbf{D}.$$

Then by Theorem 2.14 (ii), and by Proposition 4.19, the functions $\Upsilon^+[\zeta - \mathrm{id}_{\partial\mathbf{D}}]$ and $\Upsilon_\phi^-[\zeta \circ \phi^{(-1)}]$ vanish identically. In particular, we have

$$\Upsilon^+[\zeta - \mathrm{id}_{\partial\mathbf{D}}](\tau) = -\Upsilon_\phi^-[\zeta \circ \phi^{(-1)}] \circ \phi(\tau), \qquad \forall \tau \in \partial\mathbf{D}. \tag{4.29}$$

By Theorem 2.14 (ii), equation (4.29) coincides with equation (4.7). We now prove statement (ii). By adding and subtracting equation (4.6) and (4.7), we obtain that $(\mathbf{I} + \mathbf{S})[\zeta - \mathrm{id}_{\partial\mathbf{D}}] = 0$ and that $\{(\mathbf{I} - \mathbf{S}_\phi)[\zeta \circ \phi^{(-1)}]\} \circ \phi = 0$ for $\zeta \equiv \mathbf{s}[\phi]$. Thus by Theorem 2.14 (iii), (iv), F and G as in the statement exist. The uniqueness of F and G and equality $G_{|\partial\mathbf{D}} = \mathbf{s}[\phi]$ follow by Theorem 4.5 and by (i). We now prove statement (iii). By equality $F[\phi] \circ \phi = G[\phi] = \mathbf{s}[\phi]$ on $\partial\mathbf{D}$, we deduce that $F[\phi] \circ f[\phi] \circ (f[\phi]^{(-1)} \circ \phi) = G[\phi]$ on $\partial\mathbf{D}$. Then by statement (ii), we have $G[f[\phi]^{(-1)} \circ \phi] = G[\phi]$, $F[f[\phi]^{(-1)} \circ \phi] = F[\phi] \circ f[\phi]$, and thus $\mathbf{s}[\phi] = \mathbf{s}[f[\phi]^{(-1)} \circ \phi]$. We are now ready to prove that $\mathbf{s}[\phi] \in \tilde{\mathscr{T}}_{m,\alpha}$. Since $\mathbf{s}[\phi] = \mathbf{s}[f[\phi]^{(-1)} \circ \phi]$, we can assume that $\phi(\partial\mathbf{D}) = \partial\mathbf{D}$. By arguing as in Lu [22, pp. 427, 428], we prove that $F[\phi]$ and $G[\phi]$ are injective. Then by arguing as in Gakhov [6, pp. 128, 129], it follows that $F[\phi]'$ and $G[\phi]'$ do not vanish up to the boundary. Then by Lemma 2.2, we have $G[\phi]_{|\partial\mathbf{D}} = \mathbf{s}[\phi] \in \mathscr{A}_{\partial\mathbf{D}}$. Since $\mathbf{s}[\phi] = F[\phi] \circ \phi$ and $F[\phi]$ and ϕ are orientation-preserving, we also have $\mathbf{s}[\phi] \in \mathscr{A}_{\partial\mathbf{D}}^+$. By Lemma 4.3 an by (ii), we conclude that $\mathbf{s}[\phi] \in \tilde{\mathscr{T}}_{m,\alpha}$ and $G[\phi] = g[\mathbf{s}[\phi]]$. To prove equality $F[\phi] \circ f[\phi] = f[\mathbf{s}[\phi]]$, it suffices to note that both hand sides of such equality define homeomorphisms of cl $\mathbf{D}$ onto cl $I[\mathbf{s}[\phi]]$ which are holomorphic in $\mathbf{D}$ and coincide on a_1, a_2, a_3. Finally, we note that under the assumptions of statement (iv), $f[\phi]$ is the identity, and thus $F[\phi] = f[\mathbf{s}[\phi]]$ and $\mathbf{w}[\mathbf{s}[\phi]] = \phi$ by statement (iii). □

We are now ready to introduce the following.

Definition 4.30. *Let* $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0, 1[$. *We define the generalized conformal sewing operator, to be the nonlinear operator of* $C_*^{m,\alpha}(\partial\mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ *to itself which takes* ϕ *to* $\mathbf{s}[\phi]$. *We define as conformal sewing operator, the restriction of* $\mathbf{s}[\cdot]$ *to* $C_*^{m,\alpha}(\partial\mathbf{D}, \partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$.

We note that statement (iii) of Theorem 4.28 implies that the generalized conformal sewing of ϕ can be expressed as the conformal sewing of $f[\phi]^{(-1)} \circ \phi$. By Theorem 4.28 (iv), and by the definition of $\mathbf{s}[\cdot]$, we have the following.

Theorem 4.31. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let a_1, a_2, a_3 be three distinct points of $\partial \mathbf{D}$. Then the conformal welding operator $\mathbf{w}$ is a bijection of $\tilde{\mathscr{T}}_{m,\alpha}$ onto the set $\mathscr{S}_{m,\alpha} \equiv \{\phi \in C_*^{m,\alpha}(\partial \mathbf{D}, \partial \mathbf{D}) \cap \mathscr{A}_{\partial \mathbf{D}}^+ : \phi(a_j) = a_j, j = 1,2,3\}$. The inverse of the operator $\mathbf{w}_{|\tilde{\mathscr{T}}_{m,\alpha}}$ is the restriction of the generalized conformal sewing operator to $\mathscr{S}_{m,\alpha}$.*

The advantage of having defined the generalized conformal sewing operator $\mathbf{s}$ is that its domain $C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+$ is open, contrary to the domain $C_*^{m,\alpha}(\partial \mathbf{D}, \partial \mathbf{D}) \cap \mathscr{A}_{\partial \mathbf{D}}^+$ of the classical conformal sewing operator.

5. Differentiability results for the generalized conformal sewing operator

In this section we analyze the regularity of the operator $\mathbf{s}[\cdot]$ and of a related operator, which we introduce below. We first consider $\mathbf{s}[\cdot]$ by means of the following.

Theorem 5.1. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Then the generalized conformal sewing operator $\mathbf{s}$ is complex analytic from $C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+$ to $\tilde{\mathscr{T}}_{m,\alpha}$.*

Proof. Let $\tilde{\mathbf{S}}[\cdot,\cdot]$ be the operator introduced in (4.22). Let $\mathbf{\Gamma}$ be the nonlinear map of the set $(C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+) \times C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$ to the space $C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$ defined by

$$\mathbf{\Gamma}[\phi, \gamma] \equiv \gamma + \mathbf{U}_\phi[\gamma] - \mathrm{id}_{\partial \mathbf{D}} = \gamma + \frac{1}{2}\tilde{\mathbf{S}}[\mathrm{id}_{\partial \mathbf{D}}, \gamma] - \frac{1}{2}\tilde{\mathbf{S}}[\phi, \gamma] - \mathrm{id}_{\partial \mathbf{D}},$$

for all $(\phi, \gamma) \in (C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+) \times C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$. By definition of the generalized conformal sewing operator $\mathbf{s}$ and by Theorem 4.28 (i), the graph of $\mathbf{s}$ coincides with the set of zeros of $\mathbf{\Gamma}$. By [18, Prop. 4.1], which can be considered as a Schauder space version of a known result of Coifman and Meyer [3, §4], the map $\tilde{\mathbf{S}}[\cdot,\cdot]$, and thus the map $\mathbf{\Gamma}[\cdot,\cdot]$, is complex analytic. We now deduce the complex analyticity of $\mathbf{s}$ by the Implicit Function Theorem (cf. *e.g.*, Deimling [5, Thm. 15.3, p. 151].) To do so, we must show that the partial differential $d_\gamma \mathbf{\Gamma}[\phi, \mathbf{s}[\phi]]$ of $\mathbf{\Gamma}$ at $(\phi, \mathbf{s}[\phi])$ is a complex linear homeomorphism of $C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$ to itself, for all $\phi \in C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C}) \cap \mathscr{A}_{\partial \mathbf{D}}^+$. Since $\mathbf{\Gamma}$ is affine in the variable γ, we have $d_\gamma \mathbf{\Gamma}[\phi, \mathbf{s}[\phi]](\mu) = (\mathbf{I} + \mathbf{U}_\phi)[\mu]$, for all $\mu \in C_*^{m,\alpha}(\partial \mathbf{D}, \mathbf{C})$. Then by Theorem 4.27, $d_\gamma \mathbf{\Gamma}[\phi, \mathbf{s}[\phi]]$ is a complex linear homeomorphism. □

By Theorem 4.28 (ii), the function $F[\phi]_{|\partial \mathbf{D}}$ can be written as $\mathbf{s}[\phi] \circ \phi^{(-1)}$. We now turn to study the dependence of $F[\phi]_{|\partial \mathbf{D}} = \mathbf{s}[\phi] \circ \phi^{(-1)}$ upon the shift ϕ by means of the following.

Theorem 5.2. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in]0,1[$. *The (nonlinear) operator* $\mathbf{t}$ *defined by* $\mathbf{t}[\phi] \equiv \mathbf{s}[\phi] \circ \phi^{(-1)}$, *for all* $\phi \in C_*^{m,\alpha}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, *maps* $C_*^{m,\alpha}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ *to* $C_*^{m,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$, *and is continuous from* $C_*^{m,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ *to* $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$. *Let* $r \in \mathbf{N}\setminus\{0\}$, $\phi_0 \in C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$. *Then there exist an open neighborhood* $\mathscr{U}_{\phi_0}$ *of* ϕ_0 *in* $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ *and an operator* $\tilde{\mathbf{t}}_{\phi_0}$ *of class* C^r *in the real sense from* $\mathscr{U}_{\phi_0}$ *to* $C_*^{m,\alpha,0}(\partial\mathbf{D},\mathbf{C})$ *such that* $\tilde{\mathbf{t}}_{\phi_0}[\phi] = \mathbf{t}[\phi]$, *for all* $\phi \in \mathscr{U}_{\phi_0} \cap C_*^{m+r,\alpha,0}(\partial\mathbf{D},\partial\mathbf{D})$.

Proof. By Theorem 5.1, $\mathbf{s}[\cdot]$ is complex analytic from $C_*^{m+r,\alpha}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ to itself. In particular, the function $\mathbf{s}[\phi]$ is of class C_*^∞ if $\phi \in C_*^\infty(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$. Thus $\mathbf{s}[\cdot]$ is complex analytic from $C_*^{m+r,\alpha,0}(\partial\mathbf{D},\mathbf{C}) \cap \mathscr{A}_{\partial\mathbf{D}}^+$ to itself. Then we can conclude the proof by Theorem 2.9, by Theorem 2.11, and by the definition of the operator $\mathbf{t}$. □

We note that it can be actually shown that Theorem 5.2 is optimal in the frame of Schauder spaces (cf. [19, Thm. 2.17].)

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