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Tangential limits and removable sets for weighted Sobolev spaces

Dedicated to Prof. Kaoru Hatano on the occasion of his 60th birthday

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ABSTRACT. The aim in the present paper is to give a weighted version of Koskela [Ark. Mat. **37** (1999), 291–304] concerning removable sets for Sobolev functions.

1. Introduction

Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space. We will often write a point $x \in \mathbf{R}^n$ as $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}^1$. We use the notation $\mathcal{H}^\alpha$ to denote the α-dimensional Hausdorff measure.

Recently Koskela [6] studied removable sets for Sobolev spaces $W^{1,p}(\Omega)$. If $E \subset \mathbf{R}^n$ is a closed set with $\mathcal{H}^n(E) = 0$, then we say that E is removable for $W^{1,p}(\mathbf{R}^n)$ if $W^{1,p}(\mathbf{R}^n \backslash E) = W^{1,p}(\mathbf{R}^n)$ as sets.

Now suppose $E \subset \mathbf{R}^{n-1} = \{x = (x', x_n) \in \mathbf{R}^n : x_n = 0\}$. If $\mathcal{H}^{n-1}(E) = 0$, then it is clear that E is removable for $W^{1,p}(\mathbf{R}^n)$. Koskela [6] gave examples of E such that $\mathcal{H}^{n-1}(E) > 0$ and E is removable for $W^{1,p}(\mathbf{R}^n)$. Our aim in this paper is to extend his results to weighted Sobolev spaces. When E is restricted to subsets of $\mathbf{R}^{n-1}$, we consider the weights of the form $\rho(x)^\alpha$, where $\rho(x)$ denotes the distance of x from the hyperplane $\mathbf{R}^{n-1}$, that is, $\rho(x) = |x_n|$ for $x = (x', x_n) \in \mathbf{R}^n$.

For $p > 1$ and $-1 < \alpha < p - 1$, let μ_α be the Borel measure

$$d\mu_\alpha(x) = \rho(x)^\alpha dx = |x_n|^\alpha dx,$$

where $x = (x', x_n) \in \mathbf{R}^n$ and dx denotes the usual Lebesgue measure.

Let Ω be an open set in $\mathbf{R}^n$, and let $W^{1,p}(\Omega; \mu_\alpha)$ denote the weighted Sobolev space of all functions $u \in L^p(\Omega; \mu_\alpha)$ whose distributional gradient, denoted by $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, belongs to $L^p(\Omega; \mu_\alpha)$. If E is a relatively

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closed subset of Ω with $\mu_\alpha(E) = 0$, then we say that E is removable for $W^{1,p}(\Omega; \mu_\alpha)$ if

$$W^{1,p}(\Omega; \mu_\alpha) = W^{1,p}(\Omega \backslash E; \mu_\alpha)$$

as sets. As in Koskela [6], E is removable for $W^{1,p}(\Omega; \mu_\alpha)$ if and only if each function $u \in W^{1,p}(\Omega \backslash E; \mu_\alpha)$ satisfies

$$\int_{\Omega \backslash E} u \partial_j \varphi \, dx = -\int_{\Omega \backslash E} \varphi \partial_j u \, dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ and } 1 \le j \le n, \quad (1)$$

where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω.

Let $B(x, r)$ denote the n-dimensional open ball centered at x with radius r. When $1 < p < n$, we say that $E \subset \mathbf{R}^{n-1}$ is p-porous, if for $\mathcal{H}^{n-1}$-a.e. $x \in E$ there exist a sequence of positive numbers $\{r_i\}$ tending to zero, a number $q > n - p$ and a positive constant C (depending on x) such that $B(x, r_i) \cap (\mathbf{R}^{n-1} \backslash E)$ includes a set G_i of diameter R_i satisfying

(i) $\mathcal{H}_\infty^q(G_i) \ge C R_i^q$; and

(ii) $R_i \ge C r_i^{(n-1)/(n-p)}$

for all i. Here $\mathcal{H}_\infty^s(E)$ denotes the s-dimensional Hausdorff content of E, that is,

$$\mathcal{H}_\infty^s(E) = \inf \sum_{i=1}^\infty \rho_i^s,$$

where the infimum is taken over all countable covers of E by balls B_i of radius ρ_i. Notice that if E is p-porous in the sense of Koskela [6], then E is p-porous in our sense, because (i) holds for q such that $q = n - 1$ when $1 < p \le n - 1$ and $q = 1$ when $n - 1 < p < n$. In his definition of porosity, the equality $p = n - 1$ should be put in the first case to complete the proof of [6, Theorem 3.2]. For properties of p-porous sets, we refer the reader to the paper by Koskela [6].

Our main result is the following.

THEOREM 1. *Let $-1 < \alpha < p - 1$ and $1 < p < n + \alpha$. If E is $(p - \alpha)$-porous, then E is removable for $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$.*

Our result gives a weighted version of Koskela [6, Theorem A], where he considered Riesz decomposition of Sobolev functions and applied boundary limit result for p-harmonic functions. (His proof of [6, Theorem 3.2, p. 299–300] seems to use nontangential limit result instead of tangential one.) In the present paper we first study tangential limits of certain integral averages for Sobolev functions, in order to complete the proof of Theorem 1.

We also treat the case $p \geq n + \alpha$ after finishing the proof of Theorem 1. In case $p = n + \alpha$, condition (ii) of porosity will be changed to

(ii′) $R_i \geq Cr_i \exp(-r_i^{(n-1)/(1-n-\alpha)})$.

For recent work on removable singularities for quasiconformal mappings, see also Kaufman-Wu [4] and Wu [11].

Finally we would like to express our hearty thanks to the referee for his valuable comments.

2. Tangential limits of integral averages

Throughout this paper, let M denote various positive constants independent of the variables in question.

For a measurable function u on $\mathbf{R}^n$, consider the integral average over a measurable set F with respect to μ_α

$$u_F = \fint_F u \, d\mu_\alpha = \frac{1}{\mu_\alpha(F)} \int_F u \, d\mu_\alpha$$

when $0 < \mu_\alpha(F) < \infty$. For $\xi \in \mathbf{R}^{n-1}$, $\gamma \geq 1$ and $a > 0$, set

$$T_\gamma(\xi, a) = \{x \in \mathbf{R}^n_+ : |x - \xi|^\gamma < a x_n\},$$

where $\mathbf{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$. For a nonnegative measurable function f on $\mathbf{R}^n$, set

$$E_q(f) = \left\{ \xi \in \mathbf{R}^{n-1} : \limsup_{r \to 0} h_q(r)^{-1} \int_{B(\xi, r)} f(y)^p d\mu_\alpha(y) > 0 \right\},$$

where h_q is a function on $(0, \infty)$ such that

$$h_q(r) = \begin{cases} r^q & \text{if } q > 0, \\ (\log(2 + r^{-1}))^{1-p} & \text{if } q = 0. \end{cases}$$

For a nonnegative measurable function f on $\mathbf{R}^n$, we define the Riesz potential Uf by

$$Uf(x) = \int_{\mathbf{R}^n} |x - y|^{1-n} f(y) dy.$$

The second author [8, 9, 10] investigated the various boundary limits of Uf for f satisfying the weighted L^p-condition:

$$\int_{\mathbf{R}^n} f(y)^p d\mu_\alpha(y) < \infty. \tag{2}$$

We introduce the notion of the relative (p,α)-capacity $\mathrm{cap}_{p,\alpha}(\cdot\,;\Omega)$ for an open set $\Omega \subset \mathbf{R}^n$. For a compact set $K \subset \Omega$, it is defined by

$$\mathrm{cap}_{p,\alpha}(K;\Omega) = \inf \int_\Omega |\nabla u|^p d\mu_\alpha,$$

where the infimum is taken over all functions $u \in C_c^\infty(\Omega)$ such that $u \geq 1$ on K. We extend the capacity $\mathrm{cap}_{p,\alpha}(\cdot\,;\Omega)$ in the usual way (see [2, Chapter 2] and [10, Section 6.7]). We say that a set E is of (p,α)-capacity zero if

$$\mathrm{cap}_{p,\alpha}(E \cap \Omega;\Omega) = 0 \qquad \text{for every bounded open set } \Omega.$$

We say that a property holds (p,α)-quasieverywhere, often abbreviated to (p,α)-q.e., if it holds except on a set of (p,α)-capacity zero.

LEMMA 1. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (2). *Then*

$$\mathscr{H}^q(E_q(f)) = 0$$

for $q > 0$ and $\mathrm{cap}_{p,\alpha}(E_0(f)) = 0$ in case $p = n + \alpha$.

See [7, Lemma 4 and Corollary 2] for a proof of Lemma 1.

First we discuss the tangential limits of integral averages for Uf.

THEOREM 2. *Let $1 < p < \infty$, $-1 < \alpha < p - 1$ and $n - p + \alpha \geq 0$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying* (2). *For $\gamma \geq 1$, set $q = \gamma(n - p + \alpha)$. If $\xi \in \mathbf{R}^{n-1} \setminus E_q(f)$, then*

$$\lim_{x \to \xi,\, x \in T_\gamma(\xi,a)} \fint_{B(x,x_n/2)} Uf(y) d\mu_\alpha(y) = Uf(\xi)$$

for every $a > 0$.

We refer to the following technical lemma [9, Lemma 4] needed for the proof of Theorem 2.

LEMMA 2. *Let $a < 0$ and $b > -1$. If $\xi \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}^n_+$, then*

$$\int_{B(\xi,2|\xi-y|)\setminus B(y,y_n/2)} |z-y|^a |z_n|^b dz$$

$$\leq M \begin{cases} |\xi - y|^{a+b+n} + y_n^{a+b+n} & \text{when } a+b+n \neq 0, \\ \log(2|\xi - y|/y_n) & \text{when } a+b+n = 0, \end{cases}$$

where M is a positive constant independent of ξ and y.

PROOF OF THEOREM 2. We give the proof only in case $p < n + \alpha$; the case $p = n + \alpha$ is proved similarly. For $x \in \mathbf{R}^n_+$ and $y \in B(x, x_n/2)$, write

$$u_1(y) = \int_{\mathbf{R}^n \setminus B(\xi, 2|\xi - y|)} |z - y|^{1-n} f(z) dz,$$

$$u_2(y) = \int_{B(\xi, 2|\xi - y|) \setminus B(y, y_n/2)} |z - y|^{1-n} f(z) dz,$$

$$u_3(y) = \int_{B(y, y_n/2)} |z - y|^{1-n} f(z) dz.$$

If $z \in \mathbf{R}^n \setminus B(\xi, 2|\xi - y|)$, then $|z - \xi| \le 2|z - y|$, so that Lebesgue's dominated convergence theorem implies that

$$\lim_{y \to \xi} u_1(y) = Uf(\xi);$$

when $Uf(\xi) = \infty$, we can apply Fatou's lemma to obtain the above equality. Hence

$$\lim_{x \to \xi, x \in \mathbf{R}^n_+} \fint_{B(x, x_n/2)} u_1(y) d\mu_\alpha(y) = Uf(\xi).$$

By Hölder's inequality and Lemma 2 we have for $y \in B(x, x_n/2)$

$$\begin{aligned} u_2(y) &\le \left(\int_{B(\xi, 2|\xi - y|) \setminus B(y, y_n/2)} |z - y|^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right)^{1/p'} \\ &\quad \times \left(\int_{B(\xi, 2|\xi - y|)} f(z)^p |z_n|^\alpha dz \right)^{1/p} \\ &\le M \left(y_n^{p-\alpha-n} \int_{B(\xi, 2|\xi - y|)} f(z)^p d\mu_\alpha(z) \right)^{1/p}. \end{aligned}$$

Noting that $x_n/2 < y_n < 3x_n/2$ and $|\xi - x|/2 < |\xi - y| < 3|\xi - x|/2$ when $y \in B(x, x_n/2)$, we obtain

$$u_2(y) \le M \left(x_n^{p-\alpha-n} \int_{B(\xi, 3|\xi - x|)} f(z)^p d\mu_\alpha(z) \right)^{1/p}.$$

Since $\xi \notin E_q(f)$ and $x_n^{p-\alpha-n} \le M|\xi - x|^{-q}$ for $x \in T_\gamma(\xi, a)$, we see that

$$\lim_{x \to \xi, x \in T_\gamma(\xi, a)} \fint_{B(x, x_n/2)} u_2(y) d\mu_\alpha(y) = 0.$$

Finally, we have by Hölder's inequality

$$\begin{aligned}
&\fint_{B(x,x_n/2)} u_3(y)d\mu_\alpha(y)\\
&\quad\le Mx_n^{-n-\alpha}\int_{B(x,x_n/2)} y_n^\alpha\left(\int_{B(y,y_n/2)}|z-y|^{1-n}f(z)dz\right)dy\\
&\quad\le Mx_n^{-n-\alpha}\int_{\{z\in B(x,5x_n/4):z_n>x_n/4\}} f(z)\left(\int_{B(x,x_n/2)}|z-y|^{1-n}y_n^\alpha\,dy\right)dz\\
&\quad\le Mx_n^{1-n}\int_{\{z\in B(x,5x_n/4):z_n>x_n/4\}} f(z)dz\\
&\quad\le M\left(x_n^{p-n-\alpha}\int_{B(\xi,3|\xi-x|)} f(z)^p|z_n|^\alpha dz\right)^{1/p}.
\end{aligned}$$

We see that

$$\lim_{x\to\xi,\,x\in T_\gamma(\xi,a)}\fint_{B(x,x_n/2)} u_3(y)d\mu_\alpha(y)=0.$$

Now our theorem is proved.

We say that a function u is (p,α)-quasicontinuous in an open set G if for given $\varepsilon>0$ and bounded open set $D\subset G$, there exists an open set D' such that $\mathrm{cap}_{p,\alpha}(D';D)<\varepsilon$ and u is continuous as a function on $D\backslash D'$. We consider a (p,α)-quasicontinuous function v on $B(0,N)$ satisfying

$$\int_{B(0,N)}|\nabla v|^p d\mu_\alpha<\infty. \tag{3}$$

In view of [7, Corollary 1], there exists a harmonic function h on $B(0,N)$ such that

$$v(x)=\omega_n^{-1}\int_{B(0,N)}\frac{(x-y)\cdot\nabla v(y)}{|x-y|^n}\,dy+h(x) \tag{4}$$

for (p,α)-q.e. $x\in B(0,N)$, where ω_n denotes the surface measure of $\partial B(0,1)$.

For a nonnegative measurable function f on $\mathbf{R}^n$, set

$$F(f)=\left\{\xi\in\mathbf{R}^{n-1}:\int_{B(\xi,1)}|\xi-z|^{1-n}f(z)dz=\infty\right\}.$$

LEMMA 3. *Let v be a (p,α)-quasicontinuous function on $\mathbf{R}^n$ satisfying* (3) *for all $N>0$. Then*

$$\mathrm{cap}_{p,\alpha}(F(|\nabla v|))=0.$$

Further, if $q > \max\{0, n-p+\alpha\}$, *then*

$$\mathscr{H}^q(F(|\nabla v|)) = 0.$$

See [7, Lemma 4] for a proof of the above lemma.

Now, using Theorem 2 and representation (4) of Sobolev functions, we have the following result.

THEOREM 3. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and* $n-p+\alpha \geq 0$. *Let* u *be a* (p,α)*-quasicontinuous function on* $\mathbf{R}^n_+$ *satisfying*

$$\int_{\mathbf{R}^n_+ \cap B(0,N)} |\nabla u|^p d\mu_\alpha < \infty \tag{5}$$

for every $N > 0$. *For* $\gamma \geq 1$, *set* $q = \gamma(n+\alpha-p)$. *If* $\xi \in \mathbf{R}^{n-1} \backslash (F(|\nabla \bar{u}|) \cup E_q(|\nabla \bar{u}|))$, *then*

$$\lim_{x \to \xi, x \in T_\gamma(\xi,a)} \fint_{B(x, x_n/2)} u(y) d\mu_\alpha(y) \qquad \text{exists and is finite}$$

for every $a > 0$, *where* $\bar{u}$ *is a* (p,α)*-quasicontinuous extension of*

$$\bar{u}(x', x_n) = \begin{cases} u(x', x_n) & \text{for } x \in \mathbf{R}^n_+, \\ u(x', -x_n) & \text{for } x \in \mathbf{R}^n_-. \end{cases}$$

In view of Theorem 3 and Poincaré's inequality ([2]), we have the following result.

COROLLARY 1. Let $1 < p \leq n+\alpha$ and $-1 < \alpha < p-1$. If u is a harmonic function on $\mathbf{R}^n_+$ satisfying (5) for all $N > 0$, then u has a finite limit at $\xi \in \mathbf{R}^{n-1} \backslash (F(|\nabla \bar{u}|) \cup E_q(|\nabla \bar{u}|))$ along the sets $T_\gamma(\xi, a)$.

REMARK 1. If $\bar{u}$ on $B(0,N)$ is changed by the right side of (4) with $v = \bar{u} \,|\, B(0,N)$, then the limit in Theorem 3 is equal to $\bar{u}(\xi)$ for $\xi \in B(0,N) \cap \mathbf{R}^{n-1} \backslash (F(|\nabla \bar{u}|) \cup E_q(|\nabla \bar{u}|))$, where $u|A$ denotes the restriction of u to a set A.

REMARK 2. If $q > \max\{n+\alpha-p, 0\}$, then

$$\mathscr{H}^q(F(|\nabla \bar{u}|) \cup E_q(|\nabla \bar{u}|)) = 0.$$

Let $E \subset \mathbf{R}^{n-1}$ and u be a (p,α)-quasicontinuous function in $W^{1,p}(\mathbf{R}^n \backslash E; \mu_\alpha)$. Let u^+ (resp. u^-) be a (p,α)-quasicontinuous extension of $u|\mathbf{R}^n_+$ (resp. $u|\mathbf{R}^n_-$) to $\mathbf{R}^n$. Notice that E is removable for $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$ if and only if $u^+(\xi) = u^-(\xi)$ for $\mathscr{H}^{n-1}$-a.e. $\xi \in E$. Consequently, the following result, which characterizes the removable sets for $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$, can be shown by using Theorem 3.

PROPOSITION 1. *Let* $1 < p < \infty$, $-1 < \alpha < p-1$ *and* $n-p+\alpha > 0$. *Then* $E \subset \mathbf{R}^{n-1}$ *is removable for* $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$ *if and only if each* (p,α)*-quasicontinuous function* $u \in W^{1,p}(\mathbf{R}^n \backslash E; \mu_\alpha)$ *satisfies*

$$\lim_{x\to\xi,\, x\in T(\xi,a)} \fint_{B(x,\,x_n/2)} u(y)d\mu_\alpha(y) = \lim_{x\to\xi,\, \bar{x}\in T(\xi,a)} \fint_{B(x,\,-x_n/2)} u(y)d\mu_\alpha(y)$$

for $\mathcal{H}^{n-1}$*-a.e.* $\xi \in E$ *and every* $a > 0$, *where* $\bar{x} = (x', -x_n)$ *for* $x = (x', x_n)$. *Here* $T(\xi, a) = T_{(n-1)/(n+\alpha-p)}(\xi, a)$.

3. Removable sets for weighted Sobolev spaces

Our aim in this section is to prove Theorem 1. For this purpose, we further need the following result.

LEMMA 4 (cf. [3, Theorem 5.9]). *Let* A *be a subset of* $B(\xi,\rho)\cap \mathbf{R}^{n-1}$ *with* $\xi \in \mathbf{R}^{n-1}$ *and* $\rho > 0$. *If* $q > n+\alpha-p \geq 0$ *and* $\beta < 1$, *then*

$$\mathcal{H}^q_\infty(A) \leq M\rho^{q-n-\alpha+p} \int_{B(\xi,\,2\rho)} |\nabla u(x)|^p d\mu_\alpha$$

for every (p,α)*-quasicontinuous function* $u \in W^{1,p}(B(\xi,2\rho);\mu_\alpha)$ *such that* $u\,|\,A \geq 1$ *and* $u_{B(\xi,\rho)} \leq \beta$.

PROOF. Fix a (p,α)-quasicontinuous function $u \in W^{1,p}(B(\xi,2\rho);\mu_\alpha)$ with $u\,|\,A \geq 1$ and $u_{B(\xi,\rho)} \leq \beta$. Since $x \in A$ is a Lebesgue point of u except for x in a set $E \subset A$ whose Hausdorff dimension is at most $n+\alpha-p$ (see [5, Theorem 3.5] and [10, Theorem 8.2.7]) and since $q > n+\alpha-p$, we assume that each point of A is a Lebesgue point of u. First we verify Lemma 4 when there is a point $y \in A$ such that

$$|u(y) - u_{B(y,\rho)}| < (1-\beta)/2.$$

Then we have by Poincaré's inequality (see [2])

$$\begin{aligned}(1-\beta)/2 &\leq |u_{B(y,\rho)} - u_{B(\xi,\rho)}| \\ &\leq M \fint_{B(\xi,\,2\rho)} |u - u_{B(\xi,\,2\rho)}| d\mu_\alpha \leq M\rho \left(\fint_{B(\xi,\,2\rho)} |\nabla u|^p d\mu_\alpha \right)^{1/p},\end{aligned}$$

which shows that

$$\mathcal{H}^q_\infty(A) \leq \mathcal{H}^q_\infty(B(\xi,\rho)) \leq \rho^q \leq M\rho^{q-n-\alpha+p} \int_{B(\xi,\,2\rho)} |\nabla u|^p d\mu_\alpha.$$

Next we suppose that

$$|u(x) - u_{B(x,\rho)}| \geq (1-\beta)/2$$

for each point $x \in A$. Since every point of A is a Lebesgue point of u by our assumption, we have by Poincaré's inequality

$$(1-\beta)/2 \le \sum_{j=0}^{\infty} |u_{B(x,2^{-j}\rho)} - u_{B(x,2^{-j-1}\rho)}|$$

$$\le M \sum_{j=0}^{\infty} \fint_{B(x,2^{-j}\rho)} |u - u_{B(x,2^{-j}\rho)}| d\mu_\alpha$$

$$\le M \sum_{j=0}^{\infty} 2^{-j}\rho \left(\fint_{B(x,2^{-j}\rho)} |\nabla u|^p d\mu_\alpha \right)^{1/p}.$$

Since $q > n + \alpha - p$, there exists a number $j = j(x)$ such that

$$(2^{-j}\rho)^q \le M\rho^{q-n-\alpha+p} \int_{B(x,2^{-j}\rho)} |\nabla u|^p d\mu_\alpha.$$

By a covering lemma, we can find a pairwise disjoint collection $\{B(x_i, r_i)\}_{i=1}^{\infty}$ for which $x_i \in A$, $r_i = 2^{-j(x_i)}\rho$ and $A \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_i)$. Hence we see that

$$\mathscr{H}_\infty^q(A) \le \sum_{i=1}^{\infty} (5r_i)^q \le M\rho^{q-n-\alpha+p} \sum_{i=1}^{\infty} \int_{B(x_i,r_i)} |\nabla u|^p d\mu_\alpha$$

$$\le M\rho^{q-n-\alpha+p} \int_{B(\xi,2\rho)} |\nabla u|^p d\mu_\alpha,$$

as desired.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let u be a (p,α)-quasicontinuous function in $W^{1,p}(\mathbf{R}^n \backslash E; \mu_\alpha)$. Let u^+ (resp. u^-) be a (p,α)-quasicontinuous extension of $u|\mathbf{R}^n_+$ (resp. $u|\mathbf{R}^n_-$) to $\mathbf{R}^n$ as before Proposition 1. By Theorem 3 and Remark 2, there exists a set $F \subset \mathbf{R}^{n-1}$ such that $\mathscr{H}^{n-1}(F) = 0$ and

(i) $\fint_{B(x,x_n/2)} u(y) d\mu_\alpha(y)$ tends to $u^+(\xi)$ as $x \to \xi$ with $x \in T(\xi, a)$,

(ii) $\fint_{B(x,-x_n/2)} u(y) d\mu_\alpha(y)$ tends to $u^-(\xi)$ as $x \to \xi$ with $\bar{x} \in T(\xi, a)$

for each $\xi \in \mathbf{R}^{n-1} \backslash F$ and $a > 0$, where $T(\xi, a)$ as in Proposition 1. Consider the set $\tilde{E} = \{\xi \in E : u^+(\xi) \ne u^-(\xi)\}$. To complete the proof, we have only to prove

$$\mathscr{H}^{n-1}(\tilde{E}) = 0,$$

with the aid of Proposition 1. For this it suffices to show that

$$\limsup_{r \to 0} r^{1-n} \int_{B(\xi,r)} |\nabla u(x)|^p d\mu_\alpha(x) > 0 \tag{6}$$

for $\mathscr{H}^{n-1}$-a.e. $\xi \in \tilde{E} \backslash F$.

Fix $\xi \in \tilde{E} \backslash F$. Since $u^+(\xi) \ne u^-(\xi)$, by considering affine transformations, we assume that $u^+(\xi) = 0$ and $u^-(\xi) = 1$. Further, we assume that the $(p-\alpha)$-

porosity condition holds for ξ. Let r_i, q, G_i and R_i be retained from the definition of $(p-\alpha)$-porosity for E at ξ, and take $\xi_i \in \mathbf{R}^{n-1} \cap B(\xi, r_i)$ such that $G_i \subset B(\xi_i, R_i)$. We may assume that $u \geq 1/2$ in a set $A_i \subset G_i$ with $\mathscr{H}_\infty^q(A_i) \geq \mathscr{H}_\infty^q(G_i)/2$; otherwise, consider the function $1-u(x', -x_n)$.

First suppose $u^+_{B(\xi_i, R_i)} \leq 1/4$. Since $u^+ \geq 1/2$ (p,α)-q.e. on A_i and $q > n+\alpha-p$, it follows from Lemma 4 that

$$\mathscr{H}_\infty^q(A_i) \leq MR_i^{q-n-\alpha+p} \int_{B(\xi_i, 2R_i)} |\nabla u^+(x)|^p d\mu_\alpha.$$

Then, since $\mathscr{H}_\infty^q(A_i) \geq \mathscr{H}_\infty^q(G_i)/2 \geq CR_i^q/2$, we have

$$\int_{B(\xi_i, 2R_i)} |\nabla u^+(x)|^p d\mu_\alpha \geq MR_i^{n+\alpha-p}. \tag{7}$$

Next suppose $u^+_{B(\xi_i, R_i)} \geq 1/4$ for large i. Since $⨍_{B(x, x_n/2)} u(y) d\mu_\alpha(y)$ tends to zero as $x \to \xi$ along $T(\xi, a)$ for every $a > 0$, condition (ii) of $(p-\alpha)$-porosity implies that $u^+_{B(y_i, R_i)} \leq 1/8$ with $y_i = \xi_i + (0, \ldots, 0, 2R_i)$ for sufficiently large i. Then we have by Poincaré's inequality

$$\begin{aligned}
\frac{1}{8} &\leq u^+_{B(\xi_i, R_i)} - u^+_{B(y_i, R_i)} \\
&\leq M ⨍_{B(\xi_i, 3R_i)} |u^+ - u^+_{B(\xi_i, 3R_i)}| d\mu_\alpha \\
&\leq MR_i \left(⨍_{B(\xi_i, 3R_i)} |\nabla u^+(x)|^p d\mu_\alpha \right)^{1/p},
\end{aligned}$$

so that

$$\int_{B(\xi_i, 3R_i)} |\nabla u^+(x)|^p d\mu_\alpha \geq MR_i^{n+\alpha-p}. \tag{8}$$

Using (7), (8) and $R_i \leq 2r_i$, we obtain

$$\int_{B(\xi, 7r_i)} |\nabla u(x)|^p d\mu_\alpha \geq \frac{1}{2} \int_{B(\xi_i, 3R_i)} |\nabla u^+(x)|^p d\mu_\alpha \geq MR_i^{n+\alpha-p}.$$

Hence it follows from condition (ii) of $(p-\alpha)$-porosity that

$$\int_{B(\xi, 7r_i)} |\nabla u(x)|^p d\mu_\alpha \geq Mr_i^{n-1}.$$

This implies that (6) holds at ξ. Now Theorem 1 is completely proved.

REMARK 3. We can construct a $(p-\alpha)$-porous set $E \subset [0,1]^{n-1}$ such that E is not removable for $W^{1,q}(\mathbf{R}^n; \mu_\alpha)$ when $q < p$ as in Koskela [6].

4. The case $p \geq n + \alpha$

For later use, we need the following result, which can be obtained with a slight modification of the proof of Theorem 2.

PROPOSITION 2. *Let $1 < p < \infty$, $\alpha = p - n > -1$ and f be as in Theorem 2. For $\gamma > 0$, set $q = \gamma(n - 1 + \alpha)$. If $\xi \in \mathbf{R}^{n-1} \setminus E_q(f)$, then*

$$\lim_{x \to \xi, x \in V_\gamma(\xi, a)} \fint_{B(x, x_n/2)} Uf(y) d\mu_\alpha(y) = Uf(\xi)$$

for every $a > 0$, where

$$V_\gamma(\xi, a) = \{x \in \mathbf{R}^n_+ : |x - \xi| \exp(-1/|x - \xi|^\gamma) < a x_n\}.$$

PROOF. For $x \in \mathbf{R}^n_+$ and $y \in B(x, x_n/2)$, write

$$Uf(y) = u_1(y) + u_2(y) + u_3(y)$$

as in the proof of Theorem 2. In view of the estimates of u_1 and u_3 in the proof of Theorem 2, we have

$$\lim_{x \to \xi, x \in \mathbf{R}^n_+} \fint_{B(x, x_n/2)} u_1(y) d\mu_\alpha(y) = Uf(\xi)$$

and

$$\lim_{x \to \xi, x \in \mathbf{R}^n_+} \fint_{B(x, x_n/2)} u_3(y) d\mu_\alpha(y) = 0.$$

By Hölder's inequality and Lemma 2 we have for $y \in B(x, x_n/2)$

$$\begin{aligned} u_2(y) &\leq \left(\int_{B(\xi, 2|\xi - y|) \setminus B(y, y_n/2)} |z - y|^{p'(1-n)} |z_n|^{-\alpha p'/p} dz \right)^{1/p'} \\ &\quad \times \left(\int_{B(\xi, 2|\xi - y|)} f(z)^p |z_n|^\alpha dz \right)^{1/p} \\ &\leq M \left((\log(2|\xi - y|/y_n))^{p-1} \int_{B(\xi, 2|\xi - y|)} f(z)^p d\mu_\alpha(z) \right)^{1/p}. \end{aligned}$$

Hence we obtain

$$u_2(y) \leq M \left((\log(6|\xi - x|/x_n))^{p-1} \int_{B(\xi, 3|\xi - x|)} f(z)^p d\mu_\alpha(z) \right)^{1/p}.$$

Since $\xi \notin E_q(f)$ and $(\log(6|\xi - x|/x_n))^{p-1} \leq M|\xi - x|^{-q}$ for $x \in V_\gamma(\xi, a)$, we see that

$$\lim_{x\to\xi,\, x\in V_\gamma(\xi,a)} \fint_{B(x,x_n/2)} u_2(y)d\mu_\alpha(y) = 0.$$

The proof of the proposition is now completed.

Using Proposition 2 and representation (4) of Sobolev functions, we have the following results.

PROPOSITION 3. *Let $1 < p < \infty$, $\alpha = p - n > -1$ and u be as in Theorem 3. For $\gamma > 0$, set $q = \gamma(n-1+\alpha)$. If $\xi \in \mathbf{R}^{n-1}\backslash(F(|\nabla \bar{u}|) \cup E_q(|\nabla \bar{u}|))$, then*

$$\lim_{x\to\xi,\, x\in V_\gamma(\xi,a)} \fint_{B(x,x_n/2)} u(y)d\mu_\alpha(y) \qquad \text{exists and is finite}$$

for every $a > 0$.

PROPOSITION 4. *Let $1 < p < \infty$ and $\alpha = p - n > -1$. Then $E \subset \mathbf{R}^{n-1}$ is removable for $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$ if and only if each (p,α)-quasicontinuous function $u \in W^{1,p}(\mathbf{R}^n\backslash E; \mu_\alpha)$ satisfies*

$$\lim_{x\to\xi,\, x\in V(\xi,a)} \fint_{B(x,x_n/2)} u(y)d\mu_\alpha(y) = \lim_{x\to\xi,\, \bar{x}\in V(\xi,a)} \fint_{B(x,-x_n/2)} u(y)d\mu_\alpha(y)$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in E$ and every $a > 0$. Here $V(\xi,a) = V_{(n-1)/(n-1+\alpha)}(\xi,a)$.

As in the proof of Theorem 1, we obtain the following result.

THEOREM 4. *Let $1 < p < \infty$ and $\alpha = p - n > -1$. Suppose that $E \subset \mathbf{R}^{n-1}$ satisfies for $\mathcal{H}^{n-1}$-a.e. $x \in E$ there exist a sequence of positive numbers $\{r_i\}$ tending to zero, a number $q > 0$ and a positive constant C such that $B(x,r_i) \cap (\mathbf{R}^{n-1}\backslash E)$ includes a set G_i of diameter R_i satisfying*

(i) $\mathcal{H}^q_\infty(G_i) \geq CR_i^q$; *and*

(ii) $R_i \geq Cr_i \exp(-r_i^{(n-1)/(1-n-\alpha)})$

for all i. Then E is removable for $W^{1,p}(\mathbf{R}^n; \mu_\alpha)$.

PROOF. Let u be a (p,α)-quasicontinuous function in $W^{1,p}(\mathbf{R}^n\backslash E; \mu_\alpha)$. Let u^+ and u^- be as before Proposition 1. By Proposition 3 and Remark 2, there exists a set $F \subset \mathbf{R}^{n-1}$ such that $\mathcal{H}^{n-1}(F) = 0$ and

(i) $\fint_{B(x,x_n/2)} u(y)d\mu_\alpha(y)$ tends to $u^+(\xi)$ as $x \to \xi$ with $x \in V(\xi,a)$,

(ii) $\fint_{B(x,-x_n/2)} u(y)d\mu_\alpha(y)$ tends to $u^-(\xi)$ as $x \to \xi$ with $\bar{x} \in V(\xi,a)$

for each $\xi \in \mathbf{R}^{n-1}\backslash F$ and $a > 0$, where $V(\xi,a)$ as in Proposition 4. Consider the set $\tilde{E} = \{\xi \in E : u^+(\xi) \neq u^-(\xi)\}$. To complete the proof, we have only to prove

$$\mathcal{H}^{n-1}(\tilde{E}) = 0,$$

with the aid of Proposition 4. For this we show that

$$\limsup_{r\to 0}\int_{B(\xi,r)}|\nabla u(x)|^p d\mu_\alpha(x)>0 \tag{9}$$

for $\mathscr{H}^{n-1}$-a.e. $\xi\in\tilde{E}\backslash F$, which is a statement stronger than (6).

Fix $\xi\in\tilde{E}\backslash F$. Since $u^+(\xi)\neq u^-(\xi)$, by considering affine transformations, we assume that $u^+(\xi)=0$ and $u^-(\xi)=1$. Further, we assume that the condition holds for ξ. Take $\xi_i\in\mathbf{R}^{n-1}\cap B(\xi,r_i)$ such that $G_i\subset B(\xi_i,R_i)$. We may assume that $u\geq 1/2$ in a set $A_i\subset G_i$ with $\mathscr{H}^q_\infty(A_i)\geq\mathscr{H}^q_\infty(G_i)/2$; otherwise, consider the function $1-u(x',-x_n)$.

First suppose $u^+_{B(\xi_i,R_i)}\leq 1/4$. Since $u^+\geq 1/2$ (p,α)-q.e. on A_i and $q>0$, as in the proof of Theorem 1, we have

$$\int_{B(\xi_i,2R_i)}|\nabla u^+(x)|^p d\mu_\alpha\geq M. \tag{10}$$

Next suppose $u^+_{B(\xi_i,R_i)}\geq 1/4$ for large i. Since $⨍_{B(x,x_n/2)}u(y)d\mu_\alpha(y)$ tends to zero as $x\to\xi$ along $V(\xi,a)$ for every $a>0$, condition (ii) implies that $u^+_{B(y_i,R_i)}\leq 1/8$ with $y_i=\xi_i+(0,\dots,0,2R_i)$ for sufficiently large i. Then we have by Poincaré's inequality

$$\begin{aligned}\frac{1}{8}&\leq u^+_{B(\xi_i,R_i)}-u^+_{B(y_i,R_i)}\\&\leq M⨍_{B(\xi_i,3R_i)}|u^+-u^+_{B(\xi_i,3R_i)}|d\mu_\alpha\\&\leq MR_i\left(⨍_{B(\xi_i,3R_i)}|\nabla u^+(x)|^p d\mu_\alpha\right)^{1/p},\end{aligned}$$

so that

$$\int_{B(\xi_i,3R_i)}|\nabla u^+(x)|^p d\mu_\alpha\geq M. \tag{11}$$

Using (10), (11) and $R_i\leq 2r_i$, we obtain

$$\int_{B(\xi,7r_i)}|\nabla u(x)|^p d\mu_\alpha\geq\frac{1}{2}\int_{B(\xi_i,3R_i)}|\nabla u^+(x)|^p d\mu_\alpha\geq M.$$

This implies that (9) holds at ξ. Now Theorem 4 is completely proved.

Remark 4. If $E\subset\mathbf{R}^{n-1}$ has empty interior (in $\mathbf{R}^{n-1}$), then E is removable for $W^{1,p}(\mathbf{R}^n;\mu_\alpha)$ whenever $p>n+\alpha$. This immediately follows from [1, Lemma 2.5 (1)], [2, Theorem 4.12] and the next proposition.

PROPOSITION 5. *Let $u \in W^{1,p}(\mathbf{R}^n; \mu_\alpha)$ be a (p,α)-quasicontinuous function on $\mathbf{R}^n$. If $p > n+\alpha$, then $u|\mathbf{R}^{n-1}$ is Hölder continuous with exponent $(p-n-\alpha)/p$ on $\mathbf{R}^{n-1}$.*

PROOF. In this proof, we identify $x' \in \mathbf{R}^{n-1}$ with $(x',0) \in \mathbf{R}^n$. In view of [5, Theorem 3.5] or [10, Theorem 8.2.7], we see that every point of $\mathbf{R}^{n-1}$ is a Lebesgue point of u. For $x', y' \in \mathbf{R}^{n-1}$ with $r = |x'-y'|$, set $B_j(x') = B(x', 2^{-j}r)$ and $B_j(y') = B(y', 2^{-j}r)$. Using the triangle inequality and Poincaré's inequality we see that

$$\begin{aligned}
|u(x') - u_{B_0(x')}| &\le \sum_{j=0}^{\infty} |u_{B_{j+1}(x')} - u_{B_j(x')}| \\
&\le \sum_{j=0}^{\infty} \fint_{B_{j+1}(x')} |u - u_{B_j(x')}| d\mu_\alpha \\
&\le M \sum_{j=0}^{\infty} \fint_{B_j(x')} |u - u_{B_j(x')}| d\mu_\alpha \\
&\le M \sum_{j=0}^{\infty} 2^{-j} r \left(\fint_{B_j(x')} |\nabla u|^p d\mu_\alpha \right)^{1/p} \\
&\le M r^{1-(n+\alpha)/p} \sum_{j=0}^{\infty} 2^{-j(1-(n+\alpha)/p)} \left(\int_{\mathbf{R}^n} |\nabla u|^p d\mu_\alpha \right)^{1/p} \\
&\le M r^{1-(n+\alpha)/p} \left(\int_{\mathbf{R}^n} |\nabla u|^p d\mu_\alpha \right)^{1/p}.
\end{aligned}$$

In the last inequality, we used the assumption that $p > n+\alpha$. Moreover,

$$\begin{aligned}
|u_{B_0(x')} - u_{B_0(y')}| &\le |u_{B_0(x')} - u_{B_{-1}(x')}| + |u_{B_{-1}(x')} - u_{B_0(y')}| \\
&\le M \fint_{B_{-1}(x')} |u - u_{B_{-1}(x')}| d\mu_\alpha \\
&\le M r \left(\fint_{B_{-1}(x')} |\nabla u|^p d\mu_\alpha \right)^{1/p} \\
&\le M r^{1-(n+\alpha)/p} \left(\int_{\mathbf{R}^n} |\nabla u|^p d\mu_\alpha \right)^{1/p}.
\end{aligned}$$

Consequently, we have

$$|u(x') - u(y')| \leq M|x' - y'|^{1-(n+\alpha)/p} \left(\int_{\mathbf{R}^n} |\nabla u|^p d\mu_\alpha \right)^{1/p}$$

for all $x', y' \in \mathbf{R}^{n-1}$. Thus Proposition 5 is proved.

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