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On upper and lower bounds of rates of decay for nonstationary Navier-Stokes flows in the whole space

Dedicated to Professors Masayasu Mimura and Takaaki Nishida on the occasion of their 60th birthdays

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Abstract. Upper and lower bounds of rates of decay in time are studied for nonstationary Navier-Stokes flows in $\mathbf{R}^n$ with the aid of Besov spaces in which the solutions exist for all time. It is shown that there is a Besov space, with norm $\|\cdot\|$, in which the solution $u(t)$ satisfies the estimate $0 < c \le \|u(t)\| \le c'$ for all $t \ge 0$ provided the initial velocity satisfies suitable moment conditions. Our argument is then applied to the analysis of flows with cyclic symmetry, introduced by Brandolese [3], and it is shown that these flows decay more rapidly in space and time than proved in [3]. However, the existence of a lower bound as mentioned above remains open for such flows.

1. Introduction and results

This paper continues the previous works [4, 8, 9, 11] on the asymptotic behavior as $t \to \infty$ of nonstationary Navier-Stokes flows $u = (u_j)_{j=1}^n$ in $\mathbf{R}^n$, $n \ge 2$, which are governed by the integral equation:

$$\text{(IE)} \qquad u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds, \qquad t \ge 0.$$

Here, $\nabla = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $\nabla \cdot (u \otimes u) = (\sum_j \partial_j(u_j u_k))_{k=1}^n$; $u = (u_j)_{j=1}^n$ is unknown velocity and $a = (a_j)_{j=1}^n$ is a given initial velocity, both of which are required to satisfy the divergence-free condition

$$\nabla \cdot u = 0, \qquad \nabla \cdot a = 0.$$

$A = -\Delta$ is the Laplacian on $\mathbf{R}^n$, $e^{-tA}a = E_t * a$ is the convolution with the heat kernel

$$E_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t},$$

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and $P = (P_{jk})$ is the Fujita-Kato bounded projection onto the divergence-free vector fields. As shown in [4, 9], the operator $e^{-tA}P\nabla\cdot$ has the kernel function

$$F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j\partial_k\partial_\ell E_{s+t}(x)ds, \tag{1.1}$$

so that each component of the vector-valued function $e^{-tA}P\nabla\cdot(u\otimes u)$ is written as

$$[e^{-tA}P\nabla\cdot(u\otimes u)]_j = \int F_{\ell,jk}(x-y,t)(u_k u_\ell)(y)dy, \qquad j = 1,\dots,n.$$

Hereafter, we employ the summation convention and integration with respect to the spatial variables will be performed over the whole space $\mathbf{R}^n$ unless otherwise specified.

In [4, 8, 9, 10, 11] we studied asymptotic behavior as $t\to\infty$ of weak and strong solutions u in various L^q spaces, $1\le q\le\infty$, assuming that

$$\int (1+|y|)|a(y)|dy < \infty, \tag{1.2}$$

and proved, among others, the following result.

THEOREM 1.1. (i) *Let $a\in \boldsymbol{L}^n$ satisfy* (1.2) *and suppose a is small in $\boldsymbol{L}^n$ in case $n\ge 3$. Then there uniquely exists a global strong solution u of* (IE) *satisfying*

$$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)} \left\| u_j(t) + (\partial_k E_t)(\cdot)\int y_k a_j(y)dy + F_{\ell,jk}(\cdot\,,t)\int_0^\infty\int (u_k u_\ell)(y,s)dyds \right\|_q = 0 \tag{1.3}$$

for $j = 1,\dots,n$ and $1\le q\le\infty$, with $\|\cdot\|_q$ the L^q-norm. If $n\ge 3$ and if $a\in \boldsymbol{L}^2$ satisfies (1.2)*, there exists a weak solution u of* (IE) *which satisfies* (1.3) *for all $1\le q\le 2$.*

(ii) *Under the same assumptions on a as above, we have*

$$0 < c_0 \le t^{(n+2)/4}\|u(t)\|_2 \le c_1 \qquad \text{for large } t > 0$$

with appropriate constants c_0 and c_1, if and only if

$$\left(\int y_j a_m(y)dy, \int_0^\infty \int (u_k u_\ell)(y,s)dyds\right) \ne (0, c\delta_{k\ell}) \qquad \text{for all } c\ge 0. \tag{1.4}$$

(iii) *Let a satisfy* (1.2) *and*

$$|a(y)| \le C_0(1+|y|)^{-n-1}.$$

Then the corresponding strong solution u satisfies

(1.5) $$|u(x,t)| \le c_\kappa (1+|x|)^{\kappa-n-1}(1+t)^{-\kappa/2} \qquad \text{for all } 0 \le \kappa \le n+1.$$

Furthermore, if $\int |y|^n |a(y)|dy < \infty$, *then we have*

(1.6) $$\lim_{t\to\infty} t^{(n/2)(1+m/n-1/q)} \Bigg\| u_j(t) - \sum_{1\le|\alpha|\le m} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_x^\alpha E_t)(\cdot)\int y^\alpha a_j(y)dy + \sum_{|\beta|+2p\le m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p \partial_x^\beta F_{\ell,jk})(\cdot,t)\int_0^\infty \int s^p y^\beta (u_k u_\ell)(y,s)dyds \Bigg\|_q = 0$$

for $1 \le q \le \infty$, $j = 1,\dots,n$ *and* $m = 1,\dots,n$.

(iv) *Let* $n = 3,4$ *and let a satisfy* $\int (1+|y|)^{n-1}|a(y)|dy < \infty$ *and* $\int (1+|y|)^n |a(y)|^2 dy < \infty$. *Then there is a weak solution u which satisfies* (1.6) *for* $1 \le q \le 2$ *and* $m = 1,\dots,n-1$.

Assertions (i) and (iv) are proved in [4], and assertion (ii) in [11]. Assertion (iii) was proved in [4] under more stringent conditions on a as employed in [9]. We show in Section 6 that these conditions can be relaxed to the form stated in (iii). See also [20] for (1.5) with $\kappa = n+1$. Results of [9] are reproduced in [10] under weaker assumptions on initial data a.

In this paper we extend the above results to those in the following function spaces:

(1.7) $$\boldsymbol{X}^q = \boldsymbol{L}^q, \qquad 1 \le q \le \infty; \qquad \boldsymbol{X}^q = \begin{cases} \dot{\boldsymbol{B}}_{1,1}^{n(1-1/q)}, & 0 < q < 1,\ q \ne \frac{n}{n+m},\ m \in \mathbf{N}, \\ \dot{\boldsymbol{B}}_{1,\infty}^{-m}, & q = \frac{n}{n+m},\ m \in \mathbf{N}, \end{cases}$$

where $\dot{\boldsymbol{B}}_{p,q}^s$ stands for the homogeneous Besov spaces modulo polynomials ([1, 17, 18]). These spaces were employed in [8, 9] as an extrapolation of the scale of Banach spaces $\boldsymbol{L}^q$ to $q < 1$. The norm of $\boldsymbol{X}^q$ will be denoted by $\|\cdot\|_q$. Note that (see [3, 8]) since $\nabla \cdot a = 0$, condition (1.2) implies

(1.8) $$\int a(y)dy = 0, \qquad \int [y_j a_k(y) + y_k a_j(y)]dy = 0.$$

Combining (1.2) with the first condition of (1.8), we obtain (see Lemma 2.1 below)

(1.9) $$a \in \boldsymbol{X}^{n/(n+1)} \cap \boldsymbol{X}^1.$$

Furthermore, we can show that (see Lemma 2.1)

(1.10) $$\boldsymbol{X}^{n/(n+1)} \cap \boldsymbol{X}^1 \subset \boldsymbol{X}^q, \qquad \text{for all } \frac{n}{n+1} < q < 1.$$

So, the results of [8] apply to deduce the upper bounds

$$(1.11)\qquad \|u(t)\|_q \le \begin{cases} c_q(1+t)^{-(n/2)(1+1/n-1/q)} & \left(\frac{n}{n+1} \le q \le n\right), \\ c_q t^{-(n/2)(1+1/n-1/q)} & (n < q \le \infty), \end{cases}$$

for strong solutions; and

$$(1.11)'\qquad \|u(t)\|_q \le c_q(1+t)^{-(n/2)(1+1/n-1/q)} \qquad \left(\frac{n}{n+1} \le q \le 2\right),$$

for weak solutions. (1.11) is due to [20] for $q = \infty$ and $(1.11)'$ is due to [19] for $q = 2$. They were extended to $\frac{n}{n+1} \le q$ in [8]. Our first main result is the following, which extends (i) and (ii) of Theorem 1.1 to smaller values of q.

THEOREM 1.2. (i) *If a satisfies the assumptions of Theorem* 1.1 (i), *the corresponding strong solution satisfies* (1.3) *for all* $\frac{n}{n+1} \le q \le \infty$, *and the weak solution satisfies* (1.3) *for all* $\frac{n}{n+1} \le q \le 2$.

(ii) *Under the assumptions of* (i) *above, the two-sided bound*

$$(1.12)\qquad 0 < c'_q \le t^{(n/2)(1+1/n-1/q)}\|u(t)\|_q \le c_q \qquad \text{for large } t > 0,$$

is obtained with appropriate constants c_q and c'_q, if and only if a and u together satisfy (1.4). *Here,* $\frac{n}{n+1} \le q \le \infty$ *if u is a strong solution, and* $\frac{n}{n+1} \le q \le 2$ *if u is a weak solution.*

Estimate (1.12) shows in particular that

$$0 < c_0 \le \|u(t)\|_{n/(n+1)} \le c_1 \qquad \text{for large } t > 0,$$

if and only if a and u together satisfy (1.4). In particular, u is bounded and does not decay in $\boldsymbol{X}^{n/(n+1)}$ whenever $(\int y_k a_j(y)dy) \ne (0)$. We shall prove Theorem 1.2 in Section 3 after preparing necessary lemmas in Section 2. Our key result is Lemma 2.2, by which we can deduce (1.3) for $\frac{n}{n+1} \le q < 1$.

Theorem 1.2 shows that we have to consider functions which *do not* satisfy (1.4) to find a class of flows with more rapid decay property. A class of such flows has recently been found by Brandolese [3], and we next examine the decay properties of these flows in more detail, by systematically employing the spaces $\boldsymbol{X}^q$. Consider the velocity fields a satisfying

$$(1.13)\qquad \int(1+|y|)^2|a(y)|dy < \infty, \qquad \int(1+|y|)^2|a(y)|^2dy < \infty,$$

$$(1.14)\qquad a_j \text{ is odd in } y_j \text{ and is even in } y_k \text{ for all } k \ne j,$$

and

$$(1.15)\qquad a_1(y_1,\dots,y_n) = a_2(y_n,y_1,\dots,y_{n-1}) = \cdots = a_n(y_2,\dots,y_n,y_1).$$

We can directly verify that if a, u and v satisfy (1.14) and (1.15), so does the function

$$w(x,t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla \cdot (u \otimes v)(s)ds.$$

So, the standard iteration method as given in [5, 6, 7, 12] yields a weak or strong solution u satisfying (1.14), (1.15), and, moreover,

$$\int (1+|y|)^2 |u(y,t)|^2 dy \le C \qquad \text{for all } t \ge 0$$

which follows from (1.13) (see [4, Appendix] or [16]). Hereafter, up to the end of this section, by a *strong solution* we mean the solution given in Theorem 1.1 (iii); and a *weak solution* means the solution given in Theorem 1.1 (iv). Thus, our strong solutions satisfy (1.5), which was the starting point in [4] for deducing Theorem 1.1 (iii). Moreover, since $n = 3$ or $n = 4$ in Theorem 1.1 (iv), we may assume (see [6] or [12]) that our weak solutions satisfy the *strong energy inequality*

(1.16)
$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 d\tau \le \|u(s)\|_2^2 \qquad \text{for } s = 0, \text{ a.e. } s > 0 \text{ and all } t \ge s.$$

In this paper we call a solution satisfying (1.14) and (1.15) *a solution with cyclic symmetry*. Direct calculation gives

$$\int (u_k u_\ell)(y,t)dy = \lambda(t)\delta_{k\ell}, \qquad \int y_j (u_k u_\ell)(y,t)dy = 0 \tag{1.17}$$

if u is a solution with cyclic symmetry. Moreover, by (1.8) and (1.14) we get $\int y_j a_k(y)dy = 0$ for all j and k; so in view of (1.17), a and u do not satisfy (1.4). Theorem 1.2 (i) thus implies

$$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)} \|u(t)\|_q = 0$$

for $\frac{n}{n+1} \le q \le \infty$ if u is a strong solution and $\frac{n}{n+1} \le q \le 2$ if u is a weak solution. Actually, we can show more. Indeed, condition (1.8) and (1.14) together imply

$$\int y^\gamma a(y)dy = 0 \qquad \text{whenever } |\gamma| \le 2 \tag{1.18}$$

provided that a satisfies (1.13). This, together with (1.6) and (1.17), implies

$$\lim_{t\to\infty} t^{(n/2)(1+2/n-1/q)} \|u(t)\|_q = 0 \tag{1.19}$$

$$\begin{cases} \text{for all } 1 \le q \le \infty, \text{ if } u \text{ is a strong solution,} \\ \text{for all } 1 \le q \le 2, \text{ if } u \text{ is a weak solution.} \end{cases}$$

We shall extend (1.19) to the case of smaller q. Indeed, in Section 4 we prove the following, which is our second main result.

THEOREM 1.3. *Let $a \in \mathscr{S}$ satisfy $\nabla \cdot a = 0$, (1.14) and (1.15).*

(i) *There exists a strong or weak solution u with cyclic symmetry which satisfies*

$$\|u(t)\|_q \le c_q(1+t)^{-(n/2)(1+3/n-1/q)}. \tag{1.20}$$

Here, $\frac{n}{n+3} \le q \le \infty$ for a strong solution, and $\frac{n}{n+3} \le q \le 2$ for a weak solution. In particular,

$$\lim_{t\to\infty} t^{(n/2)(1+2/n-1/q)}\|u(t)\|_q = 0$$

for $\frac{n}{n+3} \le q \le \infty$ if u is a strong solution, and $\frac{n}{n+3} \le q \le 2$ if u is a weak solution.

(ii) *The strong solution u given above satisfies*

$$|u(x,t)| \le c_\kappa(1+|x|)^{\kappa-n-3}(1+t)^{-\kappa/2} \qquad \text{for all } 0 \le \kappa \le n+3. \tag{1.21}$$

(iii) *Let u be a weak or strong solution given above. Then we have*

$$t^{(n/2)(1+3/n-1/q)}\|u(t)\|_q \ge c_q > 0 \qquad \text{for large } t > 0, \tag{1.22}$$

if and only if there exists $j \in \{1,\dots,n\}$ such that, for some $t > 0$,

$$\sum_{|\alpha|=3} \frac{1}{\alpha!}(\partial_x^\alpha E_t)(x)\int y^\alpha a_j(y)dy + \sum_{|\beta|=2} \frac{1}{\beta!}(\partial_x^\beta F_{\ell,jk})(x,t)\int_0^\infty \int y^\beta (u_k u_\ell)(y,s)dyds \not\equiv 0. \tag{1.23}$$

Brandolese [3] shows the existence of a strong solution with cyclic symmetry such that

$$\begin{aligned} &\|u(t)\|_q \le c_q(1+t)^{-(n/2)(1+2/n-1/q)} \qquad \text{for all } 1 \le q \le \infty, \\ &|u(x,t)| \le c_\kappa(1+|x|)^{\kappa-n-2}(1+t)^{-\kappa/2} \qquad \text{for all } 0 \le \kappa \le n+2. \end{aligned} \tag{1.24}$$

Our (1.20) and (1.21) extend (1.24) to the case $q < 1$ with improvement. The main difference between the result of [3] and Theorem 1.3 is that [3] proves the existence of a cyclically symmetric strong solution with property (1.24), while Theorem 1.3 asserts that the weaker condition (1.5) and the cyclic symmetry together imply (1.21) provided that $a \in \mathscr{S}$.

A divergence-free vector field a satisfying the assumption of Theorem 1.3 is constructed as follows. Choose $b \in \mathscr{S}$ satisfying (1.14) and (1.15), whose Fourier transform $\hat{b}$ vanishes identically near the origin. Then the vector field $a = (a_1,\dots,a_n)$ given by

$$\hat{a}_j = (\delta_{jk} - \xi_j\xi_k/|\xi|^2)\hat{b}_k(\xi)$$

is divergence-free, belongs to $\mathscr{S}$, and satisfies (1.14), (1.15) and (1.18). To get a vector field b with properties described above, we have only to take a vector field $c \in \mathscr{S}$ satisfying (1.14) and (1.15), multiply the Fourier transform $\hat{c}$ by a smooth radial cutoff function which vanishes identically near the origin, and then take the inverse Fourier transform of the resulting vector field. Actually, the above construction gives divergence-free vector fields a such that

$$\int y^\gamma a(y)\,dy = 0 \qquad \text{for every multi-index } \gamma. \tag{1.25}$$

At the end of Section 4, we will give a criterion that ensures the existence of the constant $c_q > 0$ in (1.22) in the case where $n = 2$ or $n = 3$, assuming (1.25) for the initial velocity a. However, we do not know whether such solutions exist.

As is seen from the above discussion, study of Navier-Stokes flows with fast decay involves various cancellation properties of moments of functions. This is one reason for which the theory of the Besov spaces can be effectively applied. Indeed, it is well known in Fourier analysis that the scale of Besov spaces is suitable for treating functions with such cancellation properties. Another reason for using the spaces (1.7) is that our argument heavily relies on the properties of spatial derivatives of the heat kernel. As will be shown in Section 2, these functions are effectively treated in Besov spaces.

It is also possible to consider Navier-Stokes flows with fast decay possessing other kinds of symmetry. For example, [15] shows that if the space dimension is even, there exist solutions subject to spherical symmetry. The reader is referred to [14] for the decay properties of Navier-Stokes flows and related problems.

We conclude this paper with two appendices; in Section 5 we give a full proof of (4.6) and Section 6 deals with the existence of solutions treated in Theorem 1.1 (iii).

2. Preliminaries

We begin by proving (1.9) and (1.10). It suffices to show the following result.

Lemma 2.1. *Let* $m \in \mathbf{N}$.

(i) *Suppose* $\int (1+|y|)^m |f(y)|\,dy < \infty$ *and* $\int y^\gamma f(y)\,dy = 0$ *for every* γ *with* $|\gamma| \le m-1$. *Then* $f \in \dot{B}^{-m}_{1,\infty} \cap \dot{B}^{-m\alpha}_{1,1}$ *for all* $0 < \alpha < 1$*; and we have*

$$\begin{aligned} [f]_{-m\alpha,1,1} &\le c_{m,\alpha} \int (1+|y|)^m |f(y)|\,dy, \\ [f]_{-m,1,\infty} &\le c_m \int (1+|y|)^m |f(y)|\,dy. \end{aligned} \tag{2.1}$$

Here, $[f]_{-s,1,1}$ *and* $[f]_{-m,1,\infty}$ *denote the norms of the spaces* $\dot{B}^{-s}_{1,1}$ *and* $\dot{B}^{-m}_{1,\infty}$*, respectively.*

(ii) *We have* $\dot{B}^{-m}_{1,\infty} \cap L^1 \subset \dot{B}^{-m\alpha}_{1,1}$ *for all* $0 < \alpha < 1$*, with estimate*

$$[f]_{-m\alpha,1,1} \leq c_{m,\alpha} \|f\|_1^{1-\alpha} [f]^{\alpha}_{-m,1,\infty}. \tag{2.2}$$

(iii) *The Riesz transforms* $R = (R_1, \dots, R_n)$ *defined by*

$$(\widehat{R_k f})(\xi) = \frac{i\xi_k}{|\xi|} \hat{f}(\xi), \qquad k = 1, \dots, n, i = \sqrt{-1},$$

are bounded from $\dot{B}^{-s}_{1,1}$ *to itself and from* $\dot{B}^{-m}_{1,\infty}$ *to itself, respectively.*

Proof. (i) We fix $\psi \in \mathscr{S}$ so that

$$\operatorname{supp} \hat{\psi} \subset \{2^{-1} \leq |\xi| \leq 2\} \qquad \text{and} \qquad \hat{\psi}(\xi) \geq \frac{1}{2} \qquad \text{for } \frac{3}{5} \leq |\xi| \leq \frac{5}{3};$$

and define $\psi_j(x) = 2^{jn}\psi(2^j x)$, so that $\hat{\psi}_j(\xi) = \hat{\psi}(2^{-j}\xi)$, for $j \in \mathbf{Z}$. We may assume

$$\sum_{j \in \mathbf{Z}} \hat{\psi}_j(\xi) = 1 \qquad \text{for all } \xi \neq 0. \tag{2.3}$$

The norms of $\dot{B}^{-s}_{1,1}$ and $\dot{B}^{-m}_{1,\infty}$ are given by

$$[f]_{-s,1,1} = \sum_{j \in \mathbf{Z}} 2^{-js} \|\psi_j * f\|_1, \qquad [f]_{-m,1,\infty} = \sup_{j \in \mathbf{Z}} 2^{-jm} \|\psi_j * f\|_1.$$

See [1, 17, 18] for the details. The rescaled functions $f_\lambda(x) = f(x/\lambda)$, $\lambda > 0$, then satisfy

$$[f_\lambda]_{-s,1,1} = \lambda^{s+n}[f]_{-s,1,1}, \qquad [f_\lambda]_{-m,1,\infty} = \lambda^{m+n}[f]_{-m,1,\infty}. \tag{2.4}$$

Now, it is easy to see that

$$\sum_{j \geq 0} 2^{-jm\alpha} \|\psi_j * f\|_1 \leq \|\psi\|_1 \|f\|_1 \sum_{j \geq 0} 2^{-jm\alpha} = c_{m,\alpha} \|\psi\|_1 \|f\|_1,$$

$$\sup_{j \geq 0} 2^{-jm} \|\psi_j * f\|_1 \leq \|\psi\|_1 \|f\|_1 \left(\sup_{j \geq 0} 2^{-jm} \right) = \|\psi\|_1 \|f\|_1.$$

When $j < 0$, we apply Taylor's formula to the function $y \mapsto \psi_j(x - y)$. Due to our assumption on f, we obtain

$$(\psi_j * f)(x) = \int \psi_j(x - y) f(y) dy = \int R_{m,j}(x, y) f(y) dy,$$

where

$$R_{m,j}(x,y) = m(-2^j)^m \int_0^1 (1-\theta)^{m-1} \sum_{|\gamma|=m} \frac{1}{\gamma!} (\partial_x^\gamma \psi)_j (x - y\theta) y^\gamma \, d\theta.$$

Applying Fubini's theorem gives $\|\psi_j * f\|_1 \le c_m 2^{jm} \|\nabla^m \psi\|_1 \int |y|^m |f(y)| dy$, and so

$$\sum_{j<0} 2^{-jm\alpha} \|\psi_j * f\|_1 \le c_m \sum_{j<0} 2^{jm(1-\alpha)} \int |y|^m |f(y)| dy = c_{m,\alpha} \int |y|^m |f(y)| dy,$$

$$\sup_{j<0} 2^{-jm} \|\psi_j * f\|_1 \le c_m \int |y|^m |f(y)| dy,$$

since $\alpha < 1$. Estimate (2.1) now follows immediately. This proves (i).

(ii) Suppose $f \in \dot{B}^{-m}_{1,\infty} \cap L^1$. The estimate $\|\psi_j * f\|_1 \le \|\psi\|_1 \|f\|_1$ implies

$$\sum_{j\ge 0} 2^{-jm\alpha} \|\psi_j * f\|_1 \le c_{m,\alpha} \|\psi\|_1 \|f\|_1.$$

Since $f \in \dot{B}^{-m}_{1,\infty}$ implies $\|\psi_j * f\|_1 \le 2^{jm} [f]_{-m,1,\infty}$ for all j, and since $\alpha < 1$, it follows that

$$\sum_{j<0} 2^{-jm\alpha} \|\psi_j * f\|_1 \le [f]_{-m,1,\infty} \sum_{j<0} 2^{jm(1-\alpha)} = c_{m,\alpha} [f]_{-m,1,\infty}.$$

Hence, $[f]_{-m\alpha,1,1} \le c_{m,\alpha}(\|f\|_1 + [f]_{-m,1,\infty})$. We now insert $f_\lambda(x) = f(x/\lambda^{1/m})$, $\lambda > 0$, and apply (2.4) to get

$$[f]_{-m\alpha,1,1} \le c_{m,\alpha}(\lambda^{-\alpha} \|f\|_1 + \lambda^{1-\alpha} [f]_{-m,1,\infty})$$

with $c_{m,\alpha} > 0$ independent of $\lambda > 0$. Setting $\lambda = \|f\|_1 / [f]_{-m,1,\infty}$ gives (2.2). The proof of (ii) is complete.

(iii) For all $j \in \mathbf{Z}$, we have $(\widehat{R_k \psi_j})(\xi) = (i\xi_k/|\xi|)\hat{\psi}(2^{-j}\xi) = (\widehat{R_k \psi})(2^{-j}\xi)$ and so

$$\|R_k \psi_j\|_1 = \|(R_k \psi)_j\|_1 = \|R_k \psi\|_1.$$

Take $M_k \in C_0^\infty(\mathbf{R}^n)$ so that $M_k(\xi) \equiv 0$ near $\xi = 0$ and $M_k(\xi) = i\xi_k/|\xi|$ on supp $\hat{\psi}$. Then $\|R_k \psi\|_1 = \|\check{M}_k * \psi\|_1 \le \|\check{M}_k\|_1 \|\psi\|_1$ with $\check{M}_k$ the inverse Fourier transform of M_k, and so $\|R_k \psi_j\|_1 \le M$ with $M > 0$ independent of j and k. On the other hand, the definition of ψ_j and (2.3) together imply

$$\psi_j * f = (\psi_{j-1} + \psi_j + \psi_{j+1}) * \psi_j * f.$$

Since R_k are convolution operators, we see that

$$\|R_k\psi_j * f\|_1 \le \sum_{\ell=-1}^{1} \|(R_k\psi_{j+\ell}) * \psi_j * f\|_1 \le \sum_{\ell=-1}^{1} \|R_k\psi_{j+\ell}\|_1 \|\psi_j * f\|_1 \le 3M\|\psi_j * f\|_1.$$

This implies the desired results. The proof of Lemma 2.1 is complete.

We next consider the functions ∇E_t, $F_{\ell,jk}$ and their derivatives in the spaces $\boldsymbol{X}^q$.

Lemma 2.2. *Let* $m \in \mathbf{N}$. *Then* $\nabla^m E_t$ *and* $\nabla^{m-1} F_{\ell,jk}$ *are in* $\boldsymbol{X}^q$ *for all* $\frac{n}{n+m} \le q \le \infty$ *and satisfy*

$$\|\nabla^m E_t\|_q = c_q t^{-(n/2)(1+m/n-1/q)}, \qquad \|\nabla^{m-1} F_{\ell,jk}\|_q = c_q t^{-(n/2)(1+m/n-1/q)}.$$

Proof. The functions $\nabla^m E_t$ and $\nabla^{m-1} F_{\ell,jk}$ are of the form $t^{-(n+m)/2} K(xt^{-1/2})$ with K such that

$$K \in \boldsymbol{X}^q \qquad \text{for all } \frac{n}{n+m} \le q \le \infty. \tag{2.5}$$

This is easily checked for $\nabla^m E_t$. Indeed, in this case $K = \nabla^m E_1$ are derivatives of the rapidly decreasing function $e^{-|x|^2}$, so they belong to $\boldsymbol{X}^{n/(n+m)} \cap \boldsymbol{L}^1 \cap \boldsymbol{L}^\infty$. Lemma 2.1 (ii) ensures the desired result (2.5) for $\nabla^m E_t$. As for $\nabla^{m-1} F_{\ell,jk}$, note that $F(\cdot,1)$'s are kernel functions of the operator $P\nabla e^{-tA}$ at $t = 1$. Since $P = I + R \otimes R$ with $R = (R_1, \ldots, R_n)$ the Riesz transforms, Lemma 2.1 (iii) implies that $K = \nabla^{m-1} F(\cdot,1)$ are in $\boldsymbol{X}^{n/(n+m)}$. We easily see by direct calculation that the functions K are bounded and integrable on $\mathbf{R}^n$. So, (2.5) is deduced from Lemma 2.1 (ii). The result now follows from (2.4) and the corresponding scaling property of $\boldsymbol{L}^q$-norms. This proves Lemma 2.2.

The following lemma will be effectively applied in the subsequent sections.

Lemma 2.3. *Let* $m \in \mathbf{N}$, $\frac{n}{n+m} \le q \le \infty$, *and let* K *satisfy* (2.5). *Then,*

$$\lim_{t\to\infty} \|K(\cdot - yt^{-1/2}) - K(\cdot)\|_q = 0 \qquad \textit{for any fixed } y \in \mathbf{R}^n. \tag{2.6}$$

Proof. We use the relation

$$K(x - yt^{-1/2}) - K(x) = -t^{-1/2}\int_0^1 (y \cdot \nabla K)(x - yt^{-1/2}\tau)\,d\tau.$$

Since $\nabla K \in \boldsymbol{X}^q$ for all $\frac{n}{n+m+1} \le q \le \infty$, and since $\|\nabla K(\cdot - yt^{-1/2})\|_q = \|\nabla K\|_q$, we get

$$\|K(\cdot - yt^{-1/2}) - K(\cdot)\|_q \le t^{-1/2}|y| \int_0^1 \|\nabla K\|_q d\tau = t^{-1/2}|y|\,\|\nabla K\|_q \to 0$$

as $t \to \infty$. This shows (2.6), and the proof is complete.

3. Proof of Theorem 1.2

First we prove assertion (i). It suffices only to discuss the case $\frac{n}{n+1} \le q < 1$. Consider first the linear term $e^{-tA}a$. Since a satisfies (1.8), direct calculation gives

$$(e^{-tA}a_j)(x) = \int [E_t(x-y) - E_t(x)]a_j(y)dy = -\int_0^1 \int (\partial_k E_t)(x-y\theta)y_k a_j(y)dyd\theta,$$

so that

$$(e^{-tA}a_j)(x) + (\partial_k E_t)(x)\int y_k a_j(y)dy$$
$$= -\int_0^1 \int [(\partial_k E_t)(x-y\theta) - (\partial_k E_t)(x)]y_k a_j(y)dyd\theta.$$

By (2.4) we get, with $\frac{n}{n+1} \le q < 1$ and $K = \nabla E_1$,

$$t^{(n/2)(1+1/n-1/q)}\left\| e^{-tA}a_j + \partial_k E_t \int y_k a_j(y)dy \right\|_q$$
$$\le c\int_0^1 \int \|K(\cdot - yt^{-1/2}\theta) - K(\cdot)\|_q |y|\,|a(y)|dyd\theta.$$

Since a satisfies (1.2), applying Lemma 2.3 and the dominated convergence theorem gives

$$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)}\left\| e^{-tA}a_j + \partial_k E_t \int y_k a_j(y)dy \right\|_q = 0. \tag{3.1}$$

Consider next the nonlinear term of (IE), which is written componentwise as

$$w_j(t) \equiv -\int_0^t F_{\ell,jk}(t-s) * (u_k u_\ell)(s)ds \tag{3.2}$$
$$= -\left(\int_0^{t/2} + \int_{t/2}^t\right) F_{\ell,jk}(t-s) * (u_k u_\ell)(s)ds \equiv I_1 + I_2.$$

By (1.11) or (1.11)$'$ we have $\|u(s)\|_2^2 \le c(1+s)^{-1-n/2}$; so Lemma 2.2 implies

$$\|I_2\|_q \le \int_{t/2}^t \|F_{\ell,jk}(\cdot\,, t-s)\|_q \|u(s)\|_2^2 ds \le c\int_{t/2}^t (t-s)^{-(n/2)(1+1/n-1/q)} s^{-1-n/2}\,ds$$
$$\le ct^{-n/2-(n/2)(1+1/n-1/q)},$$

because $0 \le \frac{n}{2}\left(1+\frac{1}{n}-\frac{1}{q}\right) < \frac{1}{2}$ whenever $\frac{n}{n+1} \le q < 1$. We thus obtain

(3.3) $$t^{(n/2)(1+1/n-1/q)}\|I_2\|_q \le ct^{-n/2} \to 0 \qquad \text{as } t \to \infty.$$

We next rewrite I_1 as

$$\begin{aligned} I_1 = &-F_{\ell,jk}(x,t)\int_0^{t/2}\int (u_k u_\ell)(y,s)dyds \\ &-\int_0^{t/2}\int [F_{\ell,jk}(x-y,t-s)-F_{\ell,jk}(x,t-s)](u_k u_\ell)(y,s)dyds \\ &-\int_0^{t/2}\int [F_{\ell,jk}(x,t-s)-F_{\ell,jk}(x,t)](u_k u_\ell)(y,s)dyds, \end{aligned}$$

to obtain

(3.4) $$\begin{aligned} I_1 + F_{\ell,jk}(x,t)\int_0^{\infty}\int (u_k u_\ell)(y,s)dyds \\ = F_{\ell,jk}(x,t)\int_{t/2}^{\infty}\int (u_k u_\ell)(y,s)dyds \\ -\int_0^{t/2}\int [F_{\ell,jk}(x-y,t-s)-F_{\ell,jk}(x,t-s)](u_k u_\ell)(y,s)dyds \\ -\int_0^{t/2}\int [F_{\ell,jk}(x,t-s)-F_{\ell,jk}(x,t)](u_k u_\ell)(y,s)dyds \\ \equiv J_1 + J_2 + J_3. \end{aligned}$$

It is easy to see that

(3.5) $$t^{(n/2)(1+1/n-1/q)}\|J_1\|_q \le c\int_{t/2}^{\infty}\|u(s)\|_2^2 ds \le Ct^{-n/2} \to 0 \qquad \text{as } t \to \infty.$$

To estimate J_2, we write $F_{\ell,jk}(x,t) = t^{-(n+1)/2}K(xt^{-1/2})$ and apply Lemma 2.2 to get

(3.6) $$\begin{aligned} t^{(n/2)(1+1/n-1/q)}\|J_2\|_q \le c\int_0^{t/2}\int \|K(\cdot - y(t-s)^{-1/2}) \\ - K(\cdot)\|_q |u(y,s)|^2 dyds \\ \equiv c\int_0^{t/2}\int \varphi_t(y,s)|u(y,s)|^2 dyds. \end{aligned}$$

Observe that φ_t is bounded and Lemma 2.3 shows $\varphi_t(y,s) \to 0$ as $t \to \infty$ for fixed s and y. Therefore, the dominated convergence theorem gives

(3.7) $$\lim_{t\to\infty}\int_0^T\int \varphi_t(y,s)|u(y,s)|^2 dyds = 0 \qquad \text{for any fixed } T>0.$$

Now, given an arbitrary $\varepsilon > 0$, choose $T > 0$ so that $\int_T^\infty \|u(s)\|_2^2 ds < \varepsilon$. If $t > 2T$, then

$$\begin{aligned}\int_0^{t/2}\int \varphi_t(y,s)|u(y,s)|^2 dyds &= \left(\int_0^T + \int_T^{t/2}\right)\int \varphi_t(y,s)|u(y,s)|^2 dyds \\ &\le \int_0^T\int \varphi_t(y,s)|u(y,s)|^2 dyds + c\int_T^\infty \|u(s)\|_2^2 ds \\ &\le \int_0^T\int \varphi_t(y,s)|u(y,s)|^2 dyds + c\varepsilon\end{aligned}$$

with $c > 0$ independent of ε and t. This, together with (3.7), implies

$$\lim_{t\to\infty}\int_0^{t/2}\int \varphi_t(y,s)|u(y,s)|^2 dyds = 0$$

and so (3.6) yields

(3.8) $$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)}\|J_2\|_q = 0.$$

To estimate J_3, we write $F_{\ell,jk}(x,t) = K(x,t) = t^{-(n+1)/2}K(xt^{-1/2})$ and invoke the relation

$$K(x,t-s) - K(x,t) = -s\int_0^1 (\partial_t K)(x,t-s\theta)d\theta.$$

From (1.1) we easily see $\partial_t K = \Delta K$, which implies

$$\|\partial_t K\|_q = \|\Delta K\|_q = c_q t^{-1-(n/2)(1+1/n-1/q)} \qquad \text{for all } \frac{n}{n+1} \le q < 1.$$

Therefore, if $0 \le s \le t/2$, then

$$\begin{aligned}\|K(\cdot\,,t-s) - K(\cdot\,,t)\|_q &\le cs\int_0^1 (t-s\theta)^{-1-(n/2)(1+1/n-1/q)}d\theta \\ &\le cst^{-1-(n/2)(1+1/n-1/q)}.\end{aligned}$$

This, together with (1.11) or (1.11)$'$, implies

$$t^{(n/2)(1+1/n-1/q)}\|J_3\|_q \le ct^{-1}\int_0^{t/2} s\|u(s)\|_2^2 ds \le ct^{-1}\int_0^{t/2}(1+s)^{-n/2}ds,$$

and therefore,

$$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)}\|J_3\|_q = 0. \tag{3.9}$$

By (3.2)–(3.9), we have proved

$$\lim_{t\to\infty} t^{(n/2)(1+1/n-1/q)}\left\|w_j(t) + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q = 0$$
$$\text{for all } \frac{n}{n+1} \le q < 1.$$

Combining this with (3.1) completes the proof of assertion (i).

To prove assertion (ii), recall that condition (1.4) is equivalent to

$$\left(\partial_k E_t(x)\int y_k a_j(y)dy + F_{\ell,jk}(x,t)\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right)_{j=1}^n \not\equiv (0) \qquad \text{for all } t > 0,$$

as a function of x; see [11, Proposition 2.1]. By Lemmas 2.2 and 2.3, this is equivalent to

$$\sum_{j=1}^n \left\|\partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q = c_q t^{-(n/2)(1+1/n-1/q)} > 0. \tag{3.10}$$

Now, suppose that (1.4) holds, and so (3.10) is valid. Direct calculation then gives

$$\begin{aligned}\|u_j(t)\|_q \ge{}& \left\|\partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q \\ &- \left\|u_j(t) + \partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q.\end{aligned}$$

Here we let $t \to \infty$ and apply (1.3) to get, by (3.10),

$$\|u(t)\|_q \cong \sum_{j=1}^n \|u_j(t)\|_q \ge c_q t^{-(n/2)(1+1/n-1/q)} > 0 \qquad \text{for large } t > 0.$$

Thus, (1.11) or (1.11)$'$ implies (1.12). Conversely, suppose (1.12) holds. Since

$$\begin{aligned}&\left\|\partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q \\ &\quad\ge \|u_j(t)\|_q - \left\|u_j(t) + \partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)dyds\right\|_q,\end{aligned}$$

it follows from (1.3), (1.12) and Lemma 2.2 that

$$\sum_{j=1}^{n}\left\|\partial_k E_t \int y_k a_j(y)dy + F_{\ell,jk}\int_0^\infty \int (u_k u_\ell)(y,s)\right\|_q \equiv c_q t^{-(n/2)(1+1/n-1/q)} > 0.$$

This implies (1.4). The proof of Theorem 1.2 is complete.

4. Flows with cyclic symmetry

In this section we first prove Theorem 1.3. We then discuss the problem of finding a lower bound of rate of decay for these flows. As will be described below, it is reasonable to expect that the flows treated in Theorem 1.3 never decays in $\boldsymbol{X}^{n/(n+3)}$. However, our consideration is incomplete and so we cannot yet give a definitive answer to this question.

Recall that a strong solution means a solution given in Theorem 1.1 (iii), and a weak solution is that given in Theorem 1.1 (iv). We begin by preparing

Lemma 4.1. *Let u be a strong or weak solution with cyclic symmetry. For* $m = 0, \dots, n$,

$$\text{(4.1)} \quad \int |y|^m |u(y,s)|^2 dy \le \begin{cases} c_m(1+s)^{-1-(1+n/2)(1-m/(n+1))} & \text{if } u \text{ is a strong solution}, \\ c_m(1+s)^{-(2+n/2)(1-m/n)} & \text{if } u \text{ is a weak solution}. \end{cases}$$

Proof. By (1.19) we already know that

$$\|u(t)\|_q \le c_q(1+t)^{-(n/2)(1+2/n-1/q)} \qquad \text{for all } 1 \le q \le 2.$$

When u is a strong solution, we know $|y|^{n+1}|u(y,s)| \le C$ by (1.5). It follows that

$$\int |y|^{n+1}|u(y,s)|^2 dy \le C\int |u(y,s)|dy \le C(1+s)^{-1}.$$

Therefore, when $m = 0, 1, \dots, n, n+1$, we have

$$\int |y|^m |u(y,s)|^2 dy \le \left(\int |y|^{n+1}|u(y,s)|^2 dy\right)^{m/(n+1)} \left(\int |u(y,s)|^2 dy\right)^{1-m/(n+1)}$$
$$\le c_m(1+s)^{-1-(1+n/2)(1-m/(n+1))}.$$

When u is a weak solution, we know $\int |y|^n |u(y,s)|^2 dy \le C$ (see [4, Appendix] or [16]). Thus,

$$\int |y|^m |u(y,s)|^2 dy \le \left(\int |y|^n |u(y,s)|^2 dy\right)^{m/n} \left(\int |u(y,s)|^2\right)^{1-m/n}$$
$$\le c_m(1+s)^{-(2+n/2)(1-m/n)}$$

for $m = 0, 1, \dots, n$. This proves Lemma 4.1.

LEMMA 4.2. *For every multi-index* γ *with* $|\gamma| = m \geq 0$, *we have the estimates*

$$|\partial_x^\gamma E_t(x)| \leq c_m |x|^{-n-m}, \qquad |\partial_x^\gamma F_{\ell,jk}(x,t)| \leq c_m |x|^{-n-m-1},$$

with $c_m > 0$ *independent of* $t > 0$.

PROOF. We easily see that $|\partial_x^\gamma E_t(x)| \leq c_m t^{-(n+m)/2} e^{-c_m'|x|^2/t}$. Taking the maximum of the right-hand side with respect to $t > 0$, we get the first estimate. On the other hand, direct calculation using (1.1) gives

$$|\partial_x^\gamma F_{\ell,jk}(x,t)| \leq c_m \left(|x|^{-n-m-1} + \int_t^\infty s^{-(n+m+3)/2} e^{-c_m'|x|^2/s} \, ds \right).$$

By the change of the variable $\tau = |x|^2/s$, the last integral is estimated as

$$= |x|^{-n-m-1} \int_0^{|x|^2/t} \tau^{(n+m-1)/2} e^{-c_m'\tau} \, d\tau \leq c_m |x|^{-n-m-1}.$$

This proves Lemma 4.2.

PROOF OF THEOREM 1.3. (i) By (1.18) and Taylor's formula, we get

$$t^{(n/2)(1+3/n-1/q)} \|e^{-tA} a\|_q \leq c \sum_{|\gamma|=3} \int_0^1 \int \|(\partial_x^\gamma E_1)(\cdot - yt^{-1/2}\theta)\|_q |y|^3 |a(y)| dy d\theta.$$

Since $\|(\nabla^3 E_1)(\cdot - yt^{-1/2}\theta)\|_q = \|\nabla^3 E_1\|_q$, and since e^{-tA} defines a bounded semigroup in $\boldsymbol{X}^q$, we obtain

$$\|e^{-tA} a\|_q \leq c(1+t)^{-(n/2)(1+3/n-1/q)}. \tag{4.2}$$

We next consider

$$\begin{aligned} w_j(t) &= -\int_0^t F_{\ell,jk}(t-s) * (u_k u_\ell)(s) ds \\ &= -\left(\int_0^{t/2} + \int_{t/2}^t \right) F_{\ell,jk}(t-s) * (u_k u_\ell)(s) ds \equiv I_1 + I_2. \end{aligned}$$

By (1.17), we see that

$$\begin{aligned} I_2 = &-\sum_{k \neq \ell} \int_{t/2}^t \int [F_{\ell,jk}(x-y, t-s) - F_{\ell,jk}(x, t-s)](u_k u_\ell)(y,s) dy ds \\ &- \int_{t/2}^t \int [F_{\ell,j\ell}(x-y, t-s) - F_{\ell,j\ell}(x, t-s)] u_\ell^2(y,s) dy ds \\ &- \int_{t/2}^t \int F_{\ell,j\ell}(x, t-s) \lambda(s) ds. \end{aligned}$$

But, from (1.1) it follows that

$$F_{\ell,j\ell} = \partial_j E_t + \int_t^\infty \partial_j \Delta E_s \, ds = \partial_j E_t + \int_t^\infty \partial_s \partial_j E_s \, ds = 0, \tag{4.3}$$

and so

$$I_2 = -\int_{t/2}^t \int [F_{\ell,jk}(x-y,t-s) - F_{\ell,jk}(x,t-s)](u_k u_\ell)(y,s) dy ds.$$

Applying the second property of (1.17) gives, by Taylor's formula,

$$I_2 = -2 \sum_{|\gamma|=2} \frac{1}{\gamma!} \int_0^1 \int_{t/2}^t \int (1-\theta)(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s) y^\gamma (u_k u_\ell)(y,s) dy ds d\theta.$$

Recall that $\nabla^2 F = t^{-(n+3)/2} K(xt^{-1/2})$ with $K \in \boldsymbol{X}^q$ for all $\frac{n}{n+3} \le q \le \infty$; and (4.1) implies

$$\int |y|^2 |u(y,s)|^2 dy \le c(1+s)^{-1-\varepsilon} \tag{4.4}$$

for some $\varepsilon > 0$. Thus, for $\frac{n}{n+3} \le q < \frac{n}{n+1}$, we have

$$\|I_2\|_q \le c_q \int_{t/2}^t (t-s)^{-(n/2)(1+3/n-1/q)} (1+s)^{-1-\varepsilon} ds \le c_q (1+t)^{-(n/2)(1+3/n-1/q)-\varepsilon},$$

and $t^{(n/2)(1+3/n-1/q)} \|I_2\|_q \to 0$ as $t \to \infty$. If $q \ge 1$ and if u is a strong solution, then (1.19) gives $\|u(s)\|_q \le c_q (1+s)^{-(n/2)(1+2/n-1/q)}$. So, from

$$I_2 = -\int_{t/2}^t (F_{\ell,jk})(t-s) * (u_k u_\ell)(s) ds$$

and $\|F(t-s)\|_1 = c(t-s)^{-1/2}$, it follows that

$$\|I_2\|_q \le c_q \int_{t/2}^t \|F(t-s)\|_1 \|u(s)\|_{2q}^2 ds \le c_q \int_{t/2}^t (t-s)^{-1/2} (1+s)^{-n(1+2/n-1/(2q))} ds$$

$$\le c_q (1+t)^{1/2-n(1+2/n-1/(2q))} = c_q (1+t)^{-n/2-(n/2)(1+3/n-1/q)}.$$

Hence, $t^{(n/2)(1+3/n-1/q)} \|I_2\|_q \to 0$ as $t \to \infty$. Applying Lemma 2.1 (ii), we conclude that

$$\lim_{t\to\infty} t^{(n/2)(1+3/n-1/q)} \|I_2\|_q = 0 \tag{4.5}$$

$$\text{for all } \frac{n}{n+3} \le q \le \infty, \text{ if } u \text{ is a strong solution.}$$

When u is a weak solution, (1.19) gives $\|u(s)\|_q \le c(1+s)^{-(n/2)(1+2/n-1/q)}$ for

$1 \le q \le 2$; so the above argument shows $t^{3/2}\|I_2\|_1 \to 0$. Actually, it is possible to deduce $t^{(n/2)(1/2+3/n)}\|I_2\|_2 \to 0$, and thereby we can prove, via Lemma 2.1 (ii),

$$\lim_{t\to\infty} t^{(n/2)(1+3/n-1/q)}\|I_2\|_q = 0 \tag{4.6}$$

$$\text{for all } \frac{n}{n+3} \le q \le 2, \text{ if } u \text{ is a weak solution.}$$

Our proof of (4.6) for $q = 2$ uses (1.16) and (1.19), and it will be given in Section 5.

Similarly, the term I_1 is written as

$$\begin{aligned}
I_1 &= -2\sum_{|\gamma|=2}\frac{1}{\gamma!}\int_0^1\int_0^{t/2}\int (1-\theta)(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)y^\gamma(u_k u_\ell)(y,s)dydsd\theta \\
&= -\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x,t)\int_0^{t/2}\int y^\gamma(u_k u_\ell)(y,s)dyds \\
&\quad -\sum_{|\gamma|=2}\frac{1}{\gamma!}\int_0^{t/2}\int [(\partial_x^\gamma F_{\ell,jk})(x,t-s)-(\partial_x^\gamma F_{\ell,jk})(x,t)]y^\gamma(u_k u_\ell)(y,s)dyds \\
&\quad -2\sum_{|\gamma|=2}\frac{1}{\gamma!}\int_0^1\int_0^{t/2}\int (1-\theta)[(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)-(\partial_x^\gamma F_{\ell,jk})(x,t-s)] \\
&\quad \times y^\gamma(u_k u_\ell)(y,s)dydsd\theta \\
&\equiv -\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x,t)\int_0^{t/2}\int y^\gamma(u_k u_\ell)(y,s)dyds + J_1' + J_2',
\end{aligned}$$

so that

$$I_1 + g_t(x) = \sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x,t)\int_{t/2}^\infty\int y^\gamma(u_k u_\ell)(y,s)dyds + J_1' + J_2', \tag{4.7}$$

where

$$g_t(x) = \sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x,t)\int_0^\infty\int y^\gamma(u_k u_\ell)(y,s)dyds. \tag{4.8}$$

The integral with respect to s in (4.8) is finite, due to (4.4). We write $\nabla^2 F = t^{-(n+3)/2}K(xt^{-1/2})$, with some $K \in \boldsymbol{X}^q$, $\frac{n}{n+3} \le q \le \infty$, to obtain

$$\|g_t\|_q = c_q t^{-(n/2)(1+3/n-1/q)} \qquad \text{for all } \frac{n}{n+3} \le q \le \infty. \tag{4.9}$$

On the other hand, from (4.4) we have $\int_{t/2}^\infty\int |y|^2|u|^2dyds = O(t^{-\varepsilon})$ as $t \to \infty$, and so the first term on the right-hand side of (4.7) is $o(t^{-(n/2)(1+3/n-1/q)})$ as $t \to \infty$, for all $\frac{n}{n+3} \le q \le \infty$. For J_2', we get

$$t^{(n/2)(1+3/n-1/q)}\|J_2'\|_q \le c\int_0^1\int_0^{t/2}\int \|K(\cdot - y(t-s)^{-1/2}\theta) - K(\cdot)\|_q |y|^2 |u(y,s)|^2 dyds d\theta$$

$$\equiv \int_0^1\int_0^{t/2}\int \varphi_t(x,y,s,\theta,\eta)|y|^2|u(y,s)|^2 dydsd\theta.$$

Since $\lim_{t\to\infty} \varphi_t = 0$ by Lemma 2.3, the same argument as in the proof of Theorem 1.2 (i) gives

$$\lim_{t\to\infty} t^{(n/2)(1+3/n-1/q)}\|J_2'\|_q = 0 \qquad \text{for all } \frac{n}{n+3} \le q \le \infty.$$

We next rewrite J_1' as

$$J_1' = \sum_{|\gamma|=2}\frac{1}{\gamma!}\int_0^1\int_0^{t/2}\int (\partial_t\partial_x^\gamma F_{\ell,jk})(x,t-s\tau)sy^\gamma(u_k u_\ell)(y,s)dydsd\tau.$$

Since $\partial_t \nabla^2 F = t^{-(n+5)/2}K(xt^{-1/2})$, with $K \in \boldsymbol{X}^q$ for all $\frac{n}{n+5} \le q \le \infty$, we get by (4.4)

$$t^{(n/2)(1+3/n-1/q)}\|J_1'\|_q \le ct^{-1}\int_0^{t/2}\int s|y|^2|u(y,s)|^2 dyds \le ct^{-1}\int_0^t (1+s)^{-\varepsilon}ds \to 0$$

as $t \to \infty$. We have thus deduced

(4.10) $$\lim_{t\to\infty} t^{(n/2)(1+3/n-1/q)}\|w(t)+g_t\|_q = 0$$

$$\begin{cases} \text{for all } \frac{n}{n+3} \le q \le \infty \text{ if } u \text{ is a strong solution,} \\ \text{for all } \frac{n}{n+3} \le q \le 2 \text{ if } u \text{ is a weak solution.} \end{cases}$$

Combining (4.2), (4.9) and (4.10) gives

$$\|u(t)\|_q \le \|e^{-tA}a\|_q + \|w(t)+g_t\|_q + \|g_t\|_q \le c_q t^{-(n/2)(1+3/n-1/q)}$$

for appropriate values of q. Since $\|u(t)\|_q$ is bounded in $t > 0$, the proof of (i) is complete.

(ii) We systematically apply Lemma 4.2. Let

$$u(t) = e^{-tA}a - \int_0^t F(t-s) * (u \otimes u)(s)ds \equiv e^{-tA}a + w(t).$$

In view of (1.20) with $q = \infty$, we need only show that

$$|(e^{-tA}a)(x)| \le c(1+|x|)^{-n-3} \qquad \text{and} \qquad |w(x,t)| \le c(1+|x|)^{-n-3}$$

to deduce (1.21). Since $|u|$ and $|e^{-tA}a|$ are bounded in x and t, so is $|w(t)|$. Therefore, in what follows we always assume $|x| > 1$. Now,

$$(e^{-tA}a)(x) = \left(\int_{|y|<|x|/2} + \int_{|y|>|x|/2}\right) E_t(x-y)a(y)dy \equiv K_1 + K_2.$$

Since a is in $\mathscr{S}$,

$$|K_2| \le c \sup_{|y|>|x|/2} |a(y)| \le c(1+|x|)^{-N} \qquad \text{for all } N > 0.$$

We next invoke (1.8), (1.14), (1.15), (1.18) and Taylor's formula to get

$$\begin{aligned} K_1 &= \int_{|y|<|x|/2} \left[E_t(x-y) - \sum_{|\gamma|\le 2} \frac{(-y)^\gamma}{\gamma!} (\partial_x^\gamma E_t)(x) \right] a(y)dy \\ &\quad + \sum_{|\gamma|\le 2} \frac{(\partial_x^\gamma E_t)(x)}{\gamma!} \int_{|y|<|x|/2} (-y)^\gamma a(y)dy \\ &= 3\sum_{|\gamma|=3} \frac{1}{\gamma!} \int_0^1 \int_{|y|<|x|/2} (1-\theta)^2 (\partial_x^\gamma E_t)(x-y\theta)(-y)^\gamma a(y)dyd\theta \\ &\quad - \sum_{|\gamma|\le 2} \frac{(\partial_x^\gamma E_t)(x)}{\gamma!} \int_{|y|>|x|/2} (-y)^\gamma a(y)dy \\ &\equiv K_{11} + K_{12}. \end{aligned}$$

We easily see that

$$|K_{12}| \le c \sum_{|\gamma|\le 2} |x|^{-n-|\gamma|} \int_{|y|>|x|/2} (1+|y|)^{-N-n-1} dy \le c|x|^{-n-N}$$

for all $N > 0$; and, since $|x - y\theta| \ge |x| - |y| > |x|/2$ whenever $|y| < |x|/2$, it follows that

$$|K_{11}| \le c|x|^{-n-3} \int_{|y|<|x|/2} (1+|y|)^{-n-1} dy \le c|x|^{-n-3}.$$

Hence

$$|(e^{-tA}a)(x)| \le c(1+|x|)^{-n-3}. \tag{4.11}$$

Consider next

$$w_j(t) = -\int_0^t F_{\ell,jk}(t-s) * (u_k u_\ell)(s)ds = -\left(\int_{|y|<|x|/2} + \int_{|y|>|x|/2}\right) \equiv I_1' + I_2'.$$

Since our strong solutions are those given by Theorem 1.1 (iii), they satisfy (1.5), i.e.,

(4.12) $|u(y,s)| \le c_\kappa(1+|y|)^{\kappa-n-1}(1+s)^{-\kappa/2}$ for all $0 \le \kappa \le n+1$.

From $\|F(t-s)\|_1 = c(t-s)^{-1/2}$ and (4.12), we get

$$\begin{aligned}
|I_2'| &\le \int_0^t \int_{|y|>|x|/2} |F(x-y,t-s)|(1+|y|)^{-2n-1}(1+s)^{-1/2}dyds \\
&\le c(1+|x|)^{-2n-1}\int_0^t \int |F(x-y,t-s)|(1+s)^{-1/2}dyds \\
&\le c(1+|x|)^{-2n-1}\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds \le c(1+|x|)^{-n-3}.
\end{aligned}$$

On the other hand, we see by Taylor's formula, (1.17) and (4.3) that

$$\begin{aligned}
I_1' &= -2\int_0^1\int_0^t\int_{|y|<|x|/2}(1-\theta)\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)y^\gamma(u_k u_\ell)(y,s)dydsd\theta \\
&\quad -\int_0^t\int_{|y|<|x|/2}[F_{\ell,jk}(x,t-s)-(\partial_m F_{\ell,jk})(x,t-s)y_m](u_k u_\ell)(y,s)dyds \\
&= -2\int_0^1\int_0^t\int_{|y|<|x|/2}(1-\theta)\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)y^\gamma(u_k u_\ell)(y,s)dydsd\theta \\
&\quad +\int_0^t\int_{|y|>|x|/2}F_{\ell,jk}(x,t-s)(u_k u_\ell)(y,s)dyds \\
&\quad -\int_0^t\int_{|y|>|x|/2}(\partial_m F_{\ell,jk})(x,t-s)y_m(u_k u_\ell)(y,s)dyds \\
&\equiv I_{11}'+I_{12}'+I_{13}'.
\end{aligned}$$

By (4.4) we get

$$\begin{aligned}
|I_{11}'| &\le c|x|^{-n-3}\int_0^t\int_{|y|<|x|/2}|y|^2|u(y,s)|^2dyds \\
&\le c|x|^{-n-3}\int_0^\infty\int |y|^2|u(y,s)|^2dyds = c|x|^{-n-3}.
\end{aligned}$$

To estimate I_{12}' and I_{13}', suppose first $n \ge 3$. Direct calculation using (4.12) gives

$$|I'_{12}| \le c|x|^{-n-1} \int_0^t \int_{|y|>|x|/2} (1+|y|)^{-n-2}(1+s)^{-n/2} dyds \le c|x|^{-n-3},$$

$$|I'_{13}| \le c|x|^{-n-2} \int_0^t \int_{|y|>|x|/2} (1+|y|)^{-n-1}(1+s)^{-n/2} dyds \le c|x|^{-n-3}.$$

Collecting terms gives

$$|w(x,t)| \le |I'_1| + |I'_2| \le c(1+|x|)^{-n-3}. \tag{4.13}$$

By (4.11) and (4.13) we obtain $|u(x,t)| \le c(1+|x|)^{-n-3}$, and so (1.21) follows immediately. When $n=2$, the above argument shows $|I'_{1k}| \le c(1+|x|)^{-n-3} \cdot \log(e+t)$ for $k=2,3$, and so $|u(x,t)| \le c(1+|x|)^{-n-3}\log(e+t)$. This, together with (1.20), gives

$$|u(x,t)| \le c_\kappa (1+|x|)^{\kappa-n-3}(1+t)^{\varepsilon-\kappa/2} \qquad \text{for } 0 \le \kappa \le n+3 \text{ and } \varepsilon > 0.$$

Using this with sufficiently small $\varepsilon > 0$, we again estimate I'_{12} and I'_{13} and get

$$|I'_{12}| \le c|x|^{-n-1} \int_0^t \int_{|y|>|x|/2} (1+|y|)^{-n-2}(1+s)^{(\varepsilon-n-4)/2} dyds \le c|x|^{-n-3},$$

$$|I'_{13}| \le c|x|^{-n-2} \int_0^t \int_{|y|>|x|/2} (1+|y|)^{-n-1}(1+s)^{(\varepsilon-n-4)/2} dyds \le c|x|^{-n-3}.$$

This implies that (1.21) holds also in case $n=2$.

(iii) As in the proof of (ii), we get

$$t^{(n/2)(1+3/n-1/q)} \left\| e^{-tA}a_j + \sum_{|\alpha|=3} \frac{1}{\alpha!} \partial_x^\alpha E_t \int y^\alpha a_j(y)dy \right\|_q$$

$$\le c \sum_{|\alpha|=3} \int_0^1 \int \|(\partial_x^\alpha E_1)(\cdot - yt^{-1/2}\theta) - (\partial_x^\alpha E_1)(\cdot)\|_q |y|^3 |a_j(y)| dy d\theta \to 0$$

as $t \to \infty$. Combining this with (4.10) and (4.8), and noting that

$$\left\| \sum_{|\alpha|=3} \frac{1}{\alpha!} \partial_x^\alpha E_t \int y^\alpha a(y)dy + \sum_{|\beta|=2} \frac{1}{\beta!} (\partial_x^\beta F)(\cdot\,,t) \int_0^\infty \int y^\beta (u \otimes u)(y,s) dyds \right\|_q$$

$$\equiv c_q t^{-(n/2)(1+3/n-1/q)},$$

we can deduce (iii) as in the proof of Theorem 1.2 (ii). The proof of Theorem 1.3 is complete.

Remark. Suppose $a \in \mathscr{S}$ satisfies (1.14), (1.15) and (1.25), and so

$\|e^{-tA}a\|_q = o(t^{-M})$ for all $M > 0$ as $t \to \infty$; and let u be the corresponding strong solution treated in Theorem 1.3. Then, Theorem 1.3 (iii) shows that $t^{(n/2)(1+3/n-1/q)}\|u(t)\|_q \geq c_q > 0$ for large $t > 0$ if and only if $g_t \not\equiv 0$ for some $t > 0$, where g_t is the function given in (4.8).

Concerning this, we can show the following: consider

$$A_{jj} = \int_0^\infty \int y_j^2 u_j^2 \, dyds; \qquad B_{jk} = \int_0^\infty \int y_j^2 u_k^2 \, dyds,$$

$$D_{jk} = \int_0^\infty \int y_j u_j y_k u_k \, dyds, \qquad (j \neq k).$$

Using (1.15) for u, we easily see that $A = A_{jj}$ is independent of j. Furthermore, we always have $D_{jk} = D_{kj}$. Suppose first $n = 2$. Then $B = B_{12} = B_{21}$, $D = D_{12} = D_{21}$; and we readily see that condition $g_t \equiv 0$ implies

$$A = B + D. \tag{4.14}$$

Conversely, (4.14) implies $g_t \equiv 0$ for all $t > 0$. Suppose next $n = 3$. Due to (1.15) for u and the symmetry of D_{jk}, we always have $D \equiv D_{jk}$. Furthermore, if $g_t \equiv 0$, then $B_{jk} = B_{kj}$. Thus, (1.15) for u implies $B_{jk} \equiv B$. We can then deduce (4.14) by direct calculation. Conversely, one can directly show that if $n = 3$, if $B = B_{jk}$ are independent of j and k, and if A, B and D satisfy (4.14), then $g_t \equiv 0$ for all $t > 0$. We thus conclude that $g_t \not\equiv 0$ if and only if (4.14) breaks down, or $n = 3$ and B_{jk} is nonsymmetric.

5. Appendix A: Proof of (4.6)

We give a full proof of (4.6) in case $q = 2$ for weak solutions u given in Theorem 1.3 satisfying the strong energy inequality (1.16). The basic idea is due to [2, 5, 13]. Recall that

$$I_2 = -\int_{t/2}^t F_{\ell,jk}(t-s) * (u_k u_\ell)(s) ds.$$

Bearing this in mind, we consider for $0 < \tau < t$ the function $v = (v_1, \dots, v_n)$ with

$$v_j(t) = -\int_\tau^t F_{\ell,jk}(t-s) * (u_k u_\ell)(s) ds = u_j(t) - E_{t-\tau} * u_j(\tau).$$

The function $u_j^0(t) = E_{t-\tau} * u_j(\tau) = e^{-(t-\tau)A} u_j(\tau)$ satisfies

$$\|u^0(t)\|_2^2 + 2\int_s^t \|\nabla u^0\|_2^2 d\sigma = \|u^0(s)\|_2^2 \qquad \text{for } t \geq s \geq \tau, \tag{5.1}$$

while u satisfies (1.16), i.e.,

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u\|_2^2 d\sigma \le \|u(s)\|_2^2 \qquad \text{for a.e. } s > \tau \text{ and all } t \ge s. \tag{5.2}$$

On the other hand, since v solves weakly the initial value problem

$$\partial_t v - \Delta v = -P\nabla\cdot(u \otimes u) \qquad (t > \tau), \qquad v(\tau) = 0,$$

direct calculatuon gives

$$\langle v(t), u^\varepsilon(t)\rangle + 2\int_s^t \langle \nabla v, \nabla u^\varepsilon\rangle d\sigma = \langle v(s), u^\varepsilon(s)\rangle - \int_s^t \langle u\cdot\nabla u, u^\varepsilon\rangle d\sigma$$

for $t > s > \tau$ and $\varepsilon > 0$, where $u^\varepsilon(\sigma) = e^{-(\sigma+\varepsilon-\tau)A}u(\tau)$. Letting $\varepsilon \to 0$ yields

$$\langle v(t), u^0(t)\rangle + 2\int_s^t \langle \nabla v, \nabla u^0\rangle d\sigma = \langle v(s), u^0(s)\rangle - \int_s^t \langle u\cdot\nabla u, u^0\rangle d\sigma \tag{5.3}$$

for $t > s > \tau$. Adding (5.1) and (5.2) and then subtracting $2 \times (5.3)$, we obtain

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v\|_2^2 d\sigma \le \|v(s)\|_2^2 + 2\int_s^t \langle u\cdot\nabla u, u^0\rangle d\sigma \tag{5.4}$$

for a.e. $s > \tau$ and all $t \ge s$. But, since $\nabla\cdot u = 0$, we have $\langle u\cdot\nabla u^0, u^0\rangle = 0$, so

$$2\langle u\cdot\nabla u, u^0\rangle = 2\langle u\cdot\nabla v, u^0\rangle \le 2\|u\|_2\|\nabla v\|_2\|u^0\|_\infty \le \|\nabla v\|_2^2 + \|u\|_2^2\|u^0\|_\infty^2.$$

Inserting this in (5.4) gives

$$\|v(t)\|_2^2 + \int_s^t \|\nabla v\|_2^2 d\sigma \le \|v(s)\|_2^2 + \int_s^t \|u\|_2^2\|u^0\|_\infty^2 d\sigma \tag{5.5}$$

for a.e. $s > \tau$ and all $t \ge s$. Here we define the operators $E(\lambda)$, $\lambda > 0$, by

$$[\widehat{E(\lambda)f}](\xi) = \chi_\lambda(\xi)\hat{f}(\xi) \tag{5.6}$$

with χ_λ the indicator function of $\{\xi : |\xi|^2 \le \lambda\}$, where $\hat{f}$ is the Fourier transform of f:

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx \qquad (i = \sqrt{-1}).$$

For any $\mu > 0$, we have

$$\begin{aligned}\|\nabla v(\sigma)\|_2^2 &= \int |\xi|^2|\hat{v}(\xi,\sigma)|^2 d\xi \ge \int_{|\xi|^2\ge\mu} |\xi|^2|\hat{v}(\xi,\sigma)|^2 d\xi \\ &\ge \mu(\|v(\sigma)\|_2^2 - \|E(\mu)v(\sigma)\|_2^2),\end{aligned}$$

and so (5.5) yields

$$\|v(t)\|_2^2 + \int_s^t \mu(\sigma)\|v(\sigma)\|_2^2 d\sigma \le \|v(s)\|_2^2 + R(t,s)$$

for a.e. $s > \tau$ and all $t \ge s$, where

$$R(t,s) = \int_s^t [\mu(\sigma)\|E(\mu(\sigma))v(\sigma)\|_2^2 + \|u\|_2^2\|u^0\|_\infty^2] d\sigma.$$

Here $\mu(\sigma)$ is a positive decreasing function to be fixed later. On the other hand, since

$$v(\sigma) = -\int_\tau^\sigma e^{-(\sigma-\eta)A} P\nabla \cdot (u \otimes u)(\eta) d\eta,$$

with P and e^{-tA} defined in terms of the convolution, direct calculation gives

$$\|E(\mu)v(\sigma)\|_2 \le \int_\tau^\sigma \|E(\mu)\nabla \cdot (u \otimes u)(\eta)\|_2 d\eta \qquad \text{for fixed } \mu > 0.$$

Using (5.6) and the $L^1 - L^\infty$ estimate for the Fourier transform, we have

$$\begin{aligned} \|E(\mu)\nabla \cdot (u \otimes u)(\eta)\|_2^2 &\le \int_{|\xi|^2 \le \mu} |\xi|^2 |[\widehat{u \otimes u}](\xi,\eta)|^2 d\xi \\ &\le \|[\widehat{u \otimes u}](\eta)\|_\infty^2 \int_{|\xi|^2 \le \mu} |\xi|^2 d\xi \\ &\le c\|(u \otimes u)(\eta)\|_1^2 \mu^{1+n/2} \le c\mu^{1+n/2}\|u(\eta)\|_2^4, \end{aligned}$$

and so $\|E(\mu)v(\sigma)\|_2 \le c\mu^{(n+2)/4}\int_\tau^\sigma \|u\|_2^2 d\eta$. Therefore,

$$\mu(\sigma)\|E(\mu(\sigma))v(\sigma)\|_2^2 \le c\mu(\sigma)^{(n+4)/2}\left(\int_\tau^\sigma \|u\|_2^2 d\eta\right)^2,$$

and

$$R(t,s) \le S(t,s) \equiv c\int_s^t [\mu(\sigma)^{(n+4)/2}\left(\int_\tau^\sigma \|u\|_2^2 d\eta\right)^2 + \|u\|_2^2\|u^0\|_\infty^2] d\sigma.$$

We have thus deduced

(5.7)
$$\|v(t)\|_2^2 - S(t,s) + \int_s^t \mu(\sigma)\|v(\sigma)\|_2^2 d\sigma \le \|v(s)\|_2^2 \qquad \text{for a.e. } s > \tau \text{ and all } t \ge s.$$

It is possible to differentiate $z(s) = \int_s^t \mu(\sigma)\|v(\sigma)\|_2^2 d\sigma$ at a.e. s, to get, by (5.7),

$$z'(\sigma) = -\mu(\sigma)\|v(\sigma)\|_2^2 \le -\mu(\sigma)[z(\sigma) + \|v(t)\|_2^2 - S(t,\sigma)].$$

Here we take $\mu(\sigma) = m(\sigma-\tau)^{-1}$, $m > \frac{n}{2}+3$, and multiply both sides by $(\sigma-\tau)^m$ to get

$$(d/d\sigma)[(\sigma-\tau)^m z(\sigma)] \le -m(\sigma-\tau)^{m-1}[\|v(t)\|_2^2 - S(t,\sigma)].$$

Since $S(t,t) = 0$, integrating this by parts over $[s,t]$ gives

$$\begin{aligned}[(t-\tau)^m - (s-\tau)^m]\|v(t)\|_2^2 &\le (s-\tau)^m z(s) + \int_s^t m(\sigma-\tau)^{m-1} S(t,\sigma)d\sigma \\ &\le (s-\tau)^m z(s) - \int_s^t (\sigma-\tau)^m S'_\sigma(t,\sigma)d\sigma.\end{aligned}$$

We then pass to the limit $s \to \tau$ and divide both sides by $(t-\tau)^m$ to obtain

$$\begin{aligned}\|v(t)\|_2^2 &\le -(t-\tau)^{-m}\int_\tau^t (\sigma-\tau)^m S'_\sigma(t,\sigma)d\sigma \\ &= c_m(t-\tau)^{-m}\int_\tau^t (\sigma-\tau)^{m-n/2-2}\left(\int_\tau^\sigma \|u\|_2^2 d\eta\right)^2 d\sigma \\ &\quad + c_m(t-\tau)^{-m}\int_\tau^t (\sigma-\tau)^m \|u\|_2^2\|u^0\|_\infty^2 d\sigma \\ &\le c_m(t-\tau)^{-n/2-1}\left(\int_\tau^t \|u\|_2^2 d\sigma\right)^2 + c_m(t-\tau)^{-m}\int_\tau^t (\sigma-\tau)^m\|u\|_2^2\|u^0\|_\infty^2 d\sigma.\end{aligned}$$

Now (1.19) gives

$$\|u(\sigma)\|_2 \le c(1+\sigma)^{-1-n/4} \tag{5.8}$$

and so, since $u^0(\sigma) = e^{-(\sigma-\tau)A}u(\tau)$, the standard estimate for the heat kernel gives

$$\|u^0(\sigma)\|_\infty \le c(\sigma-\tau)^{-n/4}\|u(\tau)\|_2 \le c(\sigma-\tau)^{-n/4}(1+\tau)^{-1-n/4}.$$

Therefore,

$$\begin{aligned}\|v(t)\|_2^2 &\le c_m(t-\tau)^{-n/2-1}\left(\int_\tau^t \|u\|_2^2 d\eta\right)^2 \\ &\quad + c_m(1+\tau)^{-2-n/2}(t-\tau)^{-m}\int_\tau^t (\sigma-\tau)^{m-n/2}(1+\sigma)^{-2-n/2}d\sigma.\end{aligned}$$

Here we set $\tau = t/2$ so that $v = I_2$. Since $m > \frac{n}{2} + 3$, the above estimate gives

$$\|I_2\|_2^2 \le ct^{-n/2-1}\left(\int_{t/2}^{t} \|u\|_2^2 d\eta\right)^2 + ct^{-2-n}\int_{t/2}^{t}(1+\sigma)^{-2-n/2}d\sigma.$$

Hence (5.8) implies

$$t^{3+n/2}\|I_2\|_2^2 \le ct^2\left(\int_{t/2}^{\infty}(1+\sigma)^{-2-n/2}d\sigma\right)^2 + ct^{1-n/2}\int_{t/2}^{\infty}(1+\sigma)^{-2-n/2}d\sigma \le ct^{-n} \to 0$$

as $t \to \infty$. This proves (4.6) for $q = 2$.

6. Appendix B: On the existence of solutions given in Theorem 1.1 (iii)

The strong solutions treated in Theorem 1.1 (iii) were obtained in [9] under more stringent assumptions on a. Moreover, in deducing relevant estimates, the method of [9] employs the Hardy space theory. Here we show that such solutions are obtained under the (weaker) assumptions on a given in Theorem 1.1 (iii) without appealing to the Hardy space theory.

We begin by establishing the following

Lemma 6.1. *Let a satisfy*

$$\nabla \cdot a = 0, \qquad C_0 = \sup(1+|y|)^{n+1}|a(y)| < \infty, \qquad C_1 = \int |y|\,|a(y)|dy < \infty.$$

Then

(6.1)
$$|(e^{-tA}a)(x)| \le c(C_0 + C_1)(1+|x|)^{\kappa-n-1}(1+t)^{-\kappa/2} \qquad \text{for all } 0 \le \kappa \le n+1$$

with $c > 0$ independent of κ, C_0 and C_1.

Proof. Observe first that $|(e^{-tA}a)(x)|$ is bounded in x and $t > 0$; indeed, $|(e^{-tA}a)(x)| \le \|a\|_\infty \le C_0$. So, we assume $|x| > 1$ in estimating $|(e^{-tA}a)(x)|$ with respect to x, and $t > 1$ in estimating the same function with respect to t. Our condition on a implies (1.8), so we get

(6.2)
$$(e^{-tA}a)(x) = \int [E_t(x-y) - E_t(x)]a(y)dy = -\int_0^1 \int (y \cdot \nabla E_t)(x - y\theta)a(y)dyd\theta,$$

and

$$|(e^{-tA}a)(x)| \le ct^{-(n+1)/2}\int_0^1 \int e^{-c'|x-y\theta|^2/t}|y|\,|a(y)|dy \le cC_1 t^{-(n+1)/2}. \tag{6.3}$$

On the other hand, let

$$(e^{-tA}a)(x) = \left(\int_{|y|>|x|/2} + \int_{|y|<|x|/2}\right) E_t(x-y)a(y)dy \equiv I_1 + I_2.$$

We easily see that

$$|I_1| \le cC_0(1+|x|)^{-n-1}\int E_t(z)dz = cC_0(1+|x|)^{-n-1}. \tag{6.4}$$

Applying (1.8) yields

$$\begin{aligned} I_2 &= \int_{|y|<|x|/2} [E_t(x-y) - E_t(x)]a(y)dy + E_t(x)\int_{|y|<|x|/2} a(y)dy \\ &= -\int_0^1 \int_{|y|<|x|/2} (y\cdot\nabla E_t)(x-y\theta)a(y)dyd\theta - E_t(x)\int_{|y|>|x|/2} a(y)dy \\ &\equiv I_{21} + I_{22}. \end{aligned}$$

Direct calculation gives

$$\begin{aligned} |I_{21}| &\le ct^{-(n+1)/2}\int_{|y|<|x|/2} e^{-c'|x-y\theta|^2/t}|y|\,|a(y)|dy \le cC_1 t^{-(n+1)/2}e^{-c'|x|^2/t} \\ &\le cC_1|x|^{-n-1}, \\ |I_{22}| &\le cC_0 t^{-n/2}e^{-c'|x|^2/t}\int_{|y|>|x|/2}(1+|y|)^{-n-1}dy \le cC_0|x|^{-n}\times|x|^{-1} = cC_0|x|^{-n-1}. \end{aligned}$$

Hence $|I_2| \le c(C_0+C_1)|x|^{-n-1}$. Combining this with (6.4) gives

$$|(e^{-tA}a)(x)| \le c(C_0+C_1)|x|^{-n-1}. \tag{6.5}$$

By (6.3) and (6.5), we get (6.1). This proves Lemma 6.1.

Now that we have proved (6.1), the argument of [9] ensures the existence of a strong solution u with the initial value a satisfying (1.5), i.e.,

$$|u(x,t)| \le c_\kappa(1+|x|)^{\kappa-n-1}(1+t)^{-\kappa/2} \qquad \text{for all } 0\le\kappa\le n+1, \tag{6.6}$$

if $C_0 + C_1$ in (6.1) is sufficiently small. In [9] we proved (6.6), but the proof of [9] uses the theory of Hardy spaces in dealing with the case $\kappa = n+1$. Here we give an elementary proof of (6.6). As in [9], the crucial step is to show that if u and v satisfy (6.6), so does the function

$$w(x,t) = -\int_0^t \int F(t-s) * (u\otimes v)(s)ds.$$

From $|(u \otimes v)(y,s)| \leq c(1+|y|)^{-2n-1}(1+s)^{-1/2} \leq c(1+s)^{-1/2}$, it follows that

$$|w(x,t)| \leq \int_0^t \|F(t-s)\|_1 \|(u \otimes v)(s)\|_\infty ds \leq c \int_0^t (t-s)^{-1/2}(1+s)^{-1/2} ds \leq c,$$

which shows the boundedness of $|w(x,t)|$. So, we assume that $|x| > 1$ in estimating $|w(x,t)|$ with respect to x, and that $t > 1$ in estimating $|w(x,t)|$ with respect to t. We write

$$|w(x,t)| \leq \left(\int_0^{t/2} + \int_{t/2}^t \right) \int |F(t-s)| * |(u \otimes v)(s)| ds \equiv W_1 + W_2.$$

Direct calculation gives

$$\begin{aligned} W_1 &\leq \int_0^{t/2} \|F(t-s)\|_\infty \|u(s)\|_2 \|v(s)\|_2 ds \\ &\leq ct^{-(n+1)/2} \int_0^{t/2} (1+s)^{-1-n/2} ds \leq ct^{-(n+1)/2}, \\ W_2 &\leq c \int_{t/2}^t \|F(t-s)\|_1 \|u(s)\|_\infty \|v(s)\|_\infty ds \\ &\leq c \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-n-1} ds \leq c(1+t)^{-(n+1)/2}. \end{aligned}$$

This proves (6.6) with $\kappa = n+1$ for w. We next write

$$\begin{aligned} |w(x,t)| &\leq \int_0^t \left(\int_{|y-x|<|x|/2} + \int_{|y-x|>|x|/2} \right) |F(x-y,t-s)| \, |(u \otimes v)(y,s)| dy ds \\ &\equiv W_3 + W_4 \end{aligned}$$

to obtain

$$\begin{aligned} W_3 &\leq c \int_0^t \int_{|y-x|<|x|/2} (t-s)^{-3/4} |x-y|^{1/2-n} (1+|y|)^{-3/2-2n} (1+s)^{-1/4} dy ds \\ &\leq c(1+|x|)^{-3/2-2n} \times |x|^{1/2} \leq c(1+|x|)^{-1-2n} \leq c(1+|x|)^{-n-1}, \\ W_4 &\leq c \int_0^t \int_{|y-x|>|x|/2} |x-y|^{-n-1} (1+|y|)^{1-2n} (1+s)^{-3/2} dy ds \\ &\leq c|x|^{-n-1} \int_{|y-x|>|x|/2} (1+|y|)^{1-2n} dy \leq c|x|^{-n-1}. \end{aligned}$$

This proves (6.6) for w with $\kappa = 0$. We can now apply the fixed-point argument as in [9] to find a strong solution u of (IE) satisfying (6.6) when $C_0 + C_1$ in (6.1) is sufficiently small.

Let a strong solution u satisfy (6.6). Take $\kappa = 1$ and then $\kappa = n+1$ in (6.6), to obtain

$$\|u(s)\|_{1,w} \le c(1+s)^{-1/2}, \qquad \|u(s)\|_\infty \le c(1+s)^{-(n+1)/2},$$

where $\|\cdot\|_{1,w}$ is the quasi-norm of the weak L^1-space L^1_w. Thus, for $1 < q < \infty$,

$$\|u(s)\|_q \le c\|u(s)\|_{1,w}^{1/q}\|u(s)\|_\infty^{1-1/q} \le c(1+s)^{-(n/2)(1+1/n-1/q)}. \tag{6.7}$$

Now, (6.2) implies $\|e^{-tA}a\|_1 \le c\|\nabla E_t\|_1 \int |y|\,|a(y)|dy = ct^{-1/2}$. So (6.7) with $q = 2$ gives

$$\begin{aligned}\|u(t)\|_1 &\le ct^{-1/2} + c\int_0^t (t-s)^{-1/2}\|u(s)\|_2^2 ds \\ &\le ct^{-1/2} + c\int_0^t (t-s)^{-1/2}(1+s)^{-1-n/2} ds \le ct^{-1/2}.\end{aligned}$$

On the other hand,

$$\|u(t)\|_1 \le \|a\|_1 + c\int_0^t (t-s)^{-1/2}(1+s)^{-1-n/2} ds \le \|a\|_1 + c\int_0^t (t-s)^{-1/2} s^{-1/2}\, ds = C.$$

Therefore, $\|u(t)\|_1 \le c(1+t)^{-1/2}$. This, together with (6.6) for $\kappa = 0$, yields

$$\int |y|^{n+1}|u(y,s)|^2 dy \le (\sup|y|^{n+1}|u(y,s)|)\int |u(y,s)|dy \le c(1+s)^{-1/2}.$$

From this and (6.7) with $q = 2$, it follows via Hölder's inequality that

$$\begin{aligned}\int |y|^m|u(y,s)|^2 dy &\le \left(\int |y|^{n+1}|u(y,s)|^2 dy\right)^{m/(n+1)} \left(\int |u(y,s)|^2 dy\right)^{1-m/(n+1)} \\ &\le c(1+s)^{-1-(n-m)/2}\end{aligned}$$

for $m = 0, \dots, n+1$. This is just what we needed in [4] for deducing Theorem 1.1 (iii).

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