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Limiting behavior of attractors for systems on thin domains

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ABSTRACT. For a bounded Lipschitz domain $\Omega \subset \mathbf{R}^{M+N}$ let Ω_ε, $0 < \varepsilon$, be a family of domains squeezed in $y \in \mathbf{R}^N$ direction. On Ω_ε we consider a system of reaction-diffusion equations. We show that under certain natural conditions on the nonlinearity the generated semi-flows have global attractors which in a certain sense have limits, as $\varepsilon \downarrow 0$.

1. Introduction

Reaction-diffusion equations play an important role in a wide field of applications, as for example population ecology, neurobiology, chemical reactions, combustions, etc. For an understanding of the dynamical behavior of these equations, equilibrium solutions—or in a wider sense attractors—are especially important. The attractors depend on the shape of the underlying domain Ω. Of particular interest is squeezing Ω in one direction, getting so called thin domains. In the limit Ω collapses to a lower dimensional set, giving rise to a singular perturbation problem.

We shall show from a dynamical viewpoint that attractors (and semi-flows) of a system of reaction-diffusion equations on thin domains have a limit.

To be more precise let $\Omega \subset \mathbf{R}^{M+N}$ be a fixed smooth domain and write (x, y), $x \in \mathbf{R}^M$, $y \in \mathbf{R}^N$, for a generic point in Ω. Squeeze Ω in y-direction, i.e. for $\varepsilon > 0$ let $T_\varepsilon : \mathbf{R}^{M+N} \to \mathbf{R}^{M+N}$, $(x, y) \mapsto (x, \varepsilon y)$ and set

$$\Omega_\varepsilon := \left\{ (x, y) \in \mathbf{R}^M \times \mathbf{R}^N : \left(x, \frac{1}{\varepsilon} y \right) \in \Omega \right\} = T_\varepsilon(\Omega).$$

On Ω_ε consider the system of reaction-diffusion equations

$$V_t = \begin{pmatrix} d_1 \triangle v_1 \\ \vdots \\ d_d \triangle v_d \end{pmatrix} + f(x, y, V), \qquad t > 0, (x, y) \in \Omega_\varepsilon \tag{1.1}$$

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$$\partial_{\nu_\varepsilon} V = 0, \qquad t > 0, (x, y) \in \partial\Omega_\varepsilon, \tag{1.2}$$

where $d \in \mathbf{N}$, $d_1, \ldots, d_d \in \mathbf{R}_>$, $V = (v_1, \ldots, v_d)$, ν_ε is the outer normal to $\partial\Omega_\varepsilon$ and $f : \mathbf{R}^{M+N} \times \mathbf{R}^d \to \mathbf{R}^d$ satisfies some growth conditions to make the corresponding Nemitsky operator locally Lipschitz. We shall later impose more conditions on f to guarantee the existence of attractors $\tilde{\mathscr{A}}_\varepsilon$, and allow f to also depend linearly on the derivative $D_x V$.

It is well known that equations (1.1), (1.2) define a (local) semi-flow $\tilde{\pi}_\varepsilon$. The question arises as to what happens to these semi-flows as $\varepsilon \downarrow 0$. And, if the semi-flows $\tilde{\pi}_\varepsilon$ have global attractors $\tilde{\mathscr{A}}_\varepsilon$, how do they behave in the limit?

For scalar equation, i.e. if $d = 1$, this problem was first considered by Hale and Raugel in [7] for the case of Ω being the ordinate set of a smooth function g, i.e. if $\omega \subset \mathbf{R}^M$ is a domain and

$$\Omega = \{(x, y) \in \mathbf{R}^M \times \mathbf{R} : x \in \omega, 0 < y < g(x)\}.$$

They prove that there exists a semi-flow $\tilde{\pi}_0$ and that, in some sense, the family of attractors $(\tilde{\mathscr{A}}_\varepsilon)_{\varepsilon \geq 0}$ is upper-semi-continuous at $\varepsilon = 0$.

M. Prizzi and K. P. Rybakowski generalized this result in [9] to general Lipschitz domains $\Omega \subset \mathbf{R}^{M+N}$, which e.g. may have holes or multiple branches. The corresponding limit equation is an abstract parabolic equation defined on a subspace $H_s^1(\Omega)$ of $H^1(\Omega)$. For a wide class of domains $\Omega \subset \mathbf{R}^2$ (so called nicely decomposable domains) they described the limit problem explicitly. It is a system of second order differential equations on a graph, coupled by a compatibility condition and a Kirchoff type balance condition. They also proved—under certain natural conditions on the nonlinearity f—for a general Lipschitz domain in $\mathbf{R}^{M+N}$ the existence of the limit semi-flow $\tilde{\pi}_0$ in a strong sense, and the upper-semi-continuity of the family of attractors $(\tilde{\mathscr{A}}_\varepsilon)$. In the second paper [10] they show these attractors to be contained in inertial manifolds of finite dimension.

In general, for $N, M > 1$, there does not seem to be an explicit description of the limit problem. In [5] together with M. Prizzi we show how the limit can be characterized for some special domains, where $M = 2$, $N = 1$.

Q. Fang in [6] investigated tubular thin domains and a system of two reaction-diffusion-equations. He shows under the assumption of a positively invariant region the convergence of initial manifolds, and the relation between equilibrium solutions of (1.1), (1.2) and their limit.

In this article we generalize the results of [9] to the system (1.1) of reaction-diffusion equations on a general bounded Lipschitz domain—which includes domains with holes and multiple branches—allowing f also to depend linearly on the derivative $D_x V$.

More specifically, we show the existence of a limit semi-flow $\tilde{\pi}_0$ in a strong sense, following closely the ideas of [9]. We also show the upper-semi-continuity of the family of attractors $(\tilde{\mathscr{A}}_\varepsilon)_{0\le\varepsilon\le 1}$ at $\varepsilon = 0$, under certain natural conditions on f (see conditions H1), H2), H3) below). This is the main result of this work. Although the result is similar to that of [9]—our conditions H1) and H2) correspond to their conditions on f, and H3) is the additional condition needed for systems—the method is different. If $d = 1$, i.e. if there is only one equation, then there is a canonical Lyapunov function which can be used to prove the existence of global attractors. For systems, this is no longer true. We use a pseudo Lyapunov function to prove that the flows $\tilde{\pi}_\varepsilon$ are global and have global attractors. Moreover, our dissipativity condition (see condition H2)) is more general than a d-dimensional version of the inequality

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \le -\xi, \qquad \text{for some } \xi > 0$$

which is used in [9]. The reason is that with our more general condition H2) we can allow an (x, y)-dependence of the nonlinearity f. If f depends neither on (x, y) nor on the derivative $D_x V$, a sublinear growth of f is also allowed.

Before we can state precisely our main result, we need some notations.

Let $d, M, N \in \mathbf{N}$ be fixed numbers and $\Omega \subset \mathbf{R}^M \times \mathbf{R}^N$ be a bounded, non empty, Lipschitz domain. We shall write $(x, y) \in \Omega$, $x \in \mathbf{R}^M$, $y \in \mathbf{R}^N$ for points in Ω.

Let Ω_ε denote the squeezed domain

$$\Omega_\varepsilon := \left\{(x, y) \in \mathbf{R}^M \times \mathbf{R}^N : \left(x, \frac{1}{\varepsilon} y\right) \in \Omega\right\} = T_\varepsilon(\Omega),$$

where $T_\varepsilon : \mathbf{R}^M \times \mathbf{R}^N \to \mathbf{R}^M \times \mathbf{R}^N$, $T_\varepsilon(x, y) := (x, \varepsilon y)$.

Here, as in the whole article, unless stated otherwise, ε denotes a number in the interval $]0, 1]$.

We are interested in the behavior of the system of reaction-diffusion equations on Ω_ε given by (1.1), (1.2) as $\varepsilon \downarrow 0$.

Making a transformation onto the fixed domain Ω, (1.1), (1.2) become via $U(x, y) := V(x, \varepsilon y) = V \circ T_\varepsilon(x, y)$

$$\begin{aligned} U_t(x, y) &= \begin{pmatrix} d_1\left(\sum_{j=1}^M \partial_{x_j}^2 u_1(x, y) + \frac{1}{\varepsilon^2}\sum_{j=1}^N \partial_{y_j}^2 u_1(x, y)\right) \\ \vdots \\ d_d\left(\sum_{j=1}^M \partial_{x_j}^2 u_d(x, y) + \frac{1}{\varepsilon^2}\sum_{j=1}^N \partial_{y_j}^2 u_d(x, y)\right) \end{pmatrix} + f(x, \varepsilon y, U(x, y)) \\ &= T \triangle_\varepsilon U(x, y) + f(x, \varepsilon y, U(x, y)), \qquad t > 0, (x, y) \in \Omega \qquad (1.3) \end{aligned}$$

$$\partial_{\nu_\varepsilon} U(x, y) = 0, \qquad t > 0, (x, y) \in \partial\Omega, \tag{1.4}$$

where $d_i > 0$, $i = 1, \dots, d$, $T := diag(d_1, \dots, d_d) \in \mathbf{R}^{d\times d}$,

$$\triangle_\varepsilon := \sum_{j=1}^{M} \partial_{x_j}^2 + \frac{1}{\varepsilon^2}\sum_{j=1}^{N} \partial_{y_j}^2,$$

$$\nu_\varepsilon = \left(\nu_x, \frac{1}{\varepsilon^2}\nu_y\right),$$

and $\nu = (\nu_x, \nu_y) \in \mathbf{R}^M \times \mathbf{R}^N$ is the outer normal to $\partial\Omega$ at $(x, y) \in \partial\Omega$.

When we apply $\triangle_\varepsilon$, ∂_j (or similar operators) to a vector U, we always do so component-wise.

Note that $U \in (L^2(\Omega))^d$ (resp. $(H^1(\Omega))^d$) iff $V \in (L^2(\Omega_\varepsilon))^d$ (resp. $(H^1(\Omega_\varepsilon))^d$).

Also, (1.3), (1.4) define a flow π_ε iff (1.1), (1.2) define a corresponding flow $\tilde{\pi}_\varepsilon$. $\tilde{\pi}_\varepsilon$ has an attractor iff π_ε has one. So it is sufficient to investigate equations (1.3), (1.4).

We shall allow the nonlinearity to depend linearly on the x-derivatives of U. More precisely, we treat the following generalization of equation (1.3)

$$U_t = T\triangle_\varepsilon U + \sum_{j=1}^{M} B^j(x, \varepsilon y)\partial_j U + f(x, \varepsilon y, U), \qquad t > 0, \tag{1.5}$$

where $B^j(x, y) : \mathbf{R}^M \times \mathbf{R}^N \to \mathbf{R}^{d\times d}$ are given continuous maps, and $f \in C^1(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}^d)$.

We shall write (1.5) as an abstract equation. In order to do so, we need some notation.

For convenience we shall write L^2 instead of $L^2(\Omega)$. If the underlying set is not Ω we shall always mention it explicitly. Other functional spaces are treated likewise. Let $(.,.)_{L^2}$, $|.|_{L^2}$, $(.,.)_{H^1}$, $|.|_{H^1}$ denote the usual scalar products and norms on the Hilbert spaces L^2 and H^1, respectively. On $(L^2)^d$ and $(H^1)^d$ we define the usual scalar products and norms which will also be denoted by $(.,.)_{L^2}$, $|.|_{L^2}$, $(.,.)_{H^1}$, $|.|_{H^1}$.

The operator $\triangle_\varepsilon$ has a limit as $\varepsilon \downarrow 0$, if $\triangle_\varepsilon u$ remains bounded, that is if $\partial_{y_j} u = 0$ for all $j = 1, \dots, N$. This leads one to define

$$H_s^1 := \{u \in H^1 : \partial_{y_j} u = 0, \ \forall j = 1, \dots, N\}.$$

and L_s^2 as the closure of H_s^1 in L^2. Both H_s^1 and L_s^2 are infinite dimensional Hilbert-spaces with the usual scalar products $(.,.)_{H^1}$ and $(.,.)_{L^2}$ (see [9]).

Define the bilinear forms

$$a_\varepsilon : (H^1)^d \times (H^1)^d \to \mathbf{R}, \qquad a_\varepsilon(U,V) = \sum_{i=1}^{d} d_i \int_\Omega \nabla_\varepsilon u_i \nabla_\varepsilon v_i \, dxdy,$$

$$a_0 : (H^1_s)^d \times (H^1_s)^d \to \mathbf{R}, \qquad a_0(U,V) = \sum_{i=1}^{d} d_i \int_\Omega \nabla_x u_i \nabla_x v_i \, dxdy,$$

where ∇_ε stands for the partially weighted gradient operator

$$\nabla_\varepsilon := \begin{pmatrix} \nabla_x \\ \frac{1}{\varepsilon}\nabla_y \end{pmatrix}.$$

They generate selfadjoint operators with compact resolvents

$$A_\varepsilon : D(A_\varepsilon) \subset (H^1)^d \to (L^2)^d, \qquad A_0 : D(A_0) \subset (H^1_s)^d \to (L^2_s)^d.$$

A_ε and A_0 are sectorial.

We write equation (1.5) and boundary condition (1.4) as an abstract equation

$$U_t = -A_\varepsilon U + \hat{B}_\varepsilon U + \hat{f}_\varepsilon(U) = -A_\varepsilon U + \hat{F}_\varepsilon(U), \qquad t > 0, \tag{1.6}$$

where for $\varepsilon \in [0,1]$ we define $\hat{B}_\varepsilon, \hat{f}_\varepsilon, \hat{F}_\varepsilon : (H^1)^d \to (L^2)^d$ by

$$\hat{B}_\varepsilon U(x,y) := \sum_{j=1}^{M} B^j(x, \varepsilon y)\partial_j U(x,y), \tag{1.7}$$

$$\hat{f}_\varepsilon(U)(x,y) := f(x, \varepsilon y, U(x,y)), \tag{1.8}$$

$$\hat{F}_\varepsilon(U) := \hat{B}_\varepsilon U + \hat{f}_\varepsilon(U). \tag{1.9}$$

In the limit, equation (1.6) becomes

$$U_t = -A_0 U + \hat{F}_0(U), \qquad t > 0. \tag{1.10}$$

On f we impose the usual growth conditions, a dissipativity condition, and a technical condition needed in the case of systems (see H1), H2), H3) below). The dissipativity condition is a generalization of the more usual $U \cdot f(U) \le C - \mu|U|^p$. The latter can be interpreted geometrically as f pointing inwards on large enough circles. We generalize this concept, and assume f to point inwards on the curves $G \equiv const$, where G is a given map. These conditions guarantee the existence of global semi-flows π_ε which have global attractors $\mathscr{A}_\varepsilon$ $(0 \le \varepsilon \le 1)$.

We shall often write $U_0\pi_\varepsilon t$ and $U_0\pi_0 t$ for $\pi_\varepsilon(t, U_0)$ and $\pi_0(t, U_0)$, respectively.

The semi flows π_ε converge in a strong sense to the limit semi-flow π_0 (see

Theorem 2.2). Here strong means with respect to $|.|_\varepsilon$, an equivalent norm on $(H^1)^d$ defined by

$$|U|_\varepsilon^2 := |U|_{L^2}^2 + \sum_{i=1}^d d_i \int_\Omega \left(|\nabla_x u_i|^2 + \frac{1}{\varepsilon^2} |\nabla_y u_i|^2 \right) dxdy.$$

Our main result is the upper-semi-continuity of the attractors $\mathscr{A}_\varepsilon$:

Theorem 1.1. *Let* $\Omega \subset \mathbf{R}^M \times \mathbf{R}^N$ *be bounded, Lipschitz, and* $f \in C^1(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}^d)$ *satisfy conditions* H1), H2), H3) *below. Define the operators* $\hat{B}_\varepsilon, \hat{f}_\varepsilon, \hat{F}_\varepsilon$ *as in* (1.7), (1.8), *and* (1.9), *respectively, where we assume* $\hat{B}_0|_{(H_s^1)^d} : (H_s^1)^d \to (L_s^2)^d$.

Let A_ε, $0 \le \varepsilon \le 1$, *be as before, and* π_ε, $0 \le \varepsilon \le 1$ *be the semi-flow generated by* (1.6) *and* (1.10), *respectively.*

Then equation (1.6) *and initial condition* $U(0) = U_0 \in (H^1)^d$ *define via* $U_0\pi_\varepsilon t := U(t)$ *a global semi-flow* π_ε *on* $(H^1)^d$ *for* $0 < \varepsilon \le 1$.

Similarly, equation (1.10) *and initial condition* $U(0) = U_0 \in (H_s^1)^d$ *define via* $U_0\pi_0 t := U(t)$ *a global semi-flow* π_0 *on* $(H_s^1)^d$.

For all $0 \le \varepsilon \le 1$, *the semi-flows* π_ε *have global attractors* $\mathscr{A}_\varepsilon$ *which attract bounded sets of* $(H^1)^d$ *for* $\varepsilon > 0$, *and* $(H_s^1)^d$ *for* $\varepsilon = 0$. $\mathscr{A}_\varepsilon$ *is compact and connected in* $(H^1)^d$ *and* $(H_s^1)^d$, *respectively. Moreover,* $\mathscr{A}_\varepsilon$ *is the* ω*-limit set with respect to* π_ε

$$\mathscr{A}_\varepsilon = \omega(\{U \in (H^1)^d : |U|_\varepsilon \le \delta_f\}) \subset \{U \in (H^1)^d : |U|_\varepsilon \le \delta_f\},$$

for $\varepsilon > 0$, *and*

$$\mathscr{A}_0 = \omega(\{U \in (H_s^1)^d : |U|_{H^1} \le \tilde{\delta}_f\}) \subset \{U \in (H_s^1)^d : |U|_{H^1} \le \tilde{\delta}_f\}$$

for $\varepsilon = 0$. *Here* δ_f *is as in Theorem* 4.1, $\tilde{\delta}_f = (\min(1, d_1, \dots, d_d))^{-1/2}\delta_f$.

The family of attractors $\mathscr{A}_\varepsilon$ *is upper-semi-continuous at* $\varepsilon = 0$ *with respect to the family of norms* $|.|_\varepsilon$, *i.e.*

$$\lim_{\varepsilon \downarrow 0} \sup_{U \in \mathscr{A}_\varepsilon} \inf_{V \in \mathscr{A}_0} |U - V|_\varepsilon = 0.$$

Theorem 1.1 will be proven in §4.

The conditions we impose on f are as follows.

Let $C_\Omega > 0$ be a constant, such that $\bar{\Omega} \subset \{(x, y) \in \mathbf{R}^M \times \mathbf{R}^N : |(x, y)| < C_\Omega\}$.

H1)

$$\|D_u f(x, y, U)\| \le C_f(1 + |U|^{p_0}), \qquad \forall |(x, y)| \le C_\Omega, U \in \mathbf{R}^d,$$

$$\|D_{(x,y)} f(x, y, U)\| \le C_f(1 + |U|^{p_0+1}), \qquad \forall |(x, y)| \le C_\Omega, U \in \mathbf{R}^d,$$

where p_0, $C_f \ge 0$, and if $M + N > 2$, then $p_0 < 2/(M + N - 2)$.

H2) For all $\varepsilon \in [0,1]$

$$\nabla_u G(x,y,U) \cdot f(x,\varepsilon y,U) \le C_f - \mu_0 |U|^{p_1}, \qquad \forall |(x,y)| \le C_\Omega, U \in \mathbf{R}^d,$$

where $\mu_0 > 0$, $C_f \ge 0$, and $p_1 > 2$.

Additionally, $p_1 \ge 2(p_0+1)$, if f depends explicitly on (x,y), and $p_1 \ge 2(p_2+1)$, p_2 as in C2) below, if G depends on (x,y) or if $\hat{B}_\varepsilon \ne 0$.

H3)

$$\sum_{i,l=1}^{d} \xi_i \partial_{u_l} f_i(x,y,U) \xi_l \le C_f |\xi|^2, \qquad \forall |(x,y)| \le C_\Omega, U, \xi \in \mathbf{R}^d,$$

where $C_f \ge 0$.

Here the map G has to satisfy some conditions: it has to have a minimal growth, which allows to compare it with $|U|^2$, there are some growth conditions to ensure the existence of certain Nemitsky operators, and there is a technical condition somewhat like H3). The precise conditions we impose on $G \in C^2(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}_\ge)$ are as follows.

C1)

$$G(x,y,U) \cdot |U|^{-2} \ge \nu_0, \qquad \forall |(x,y)| \le C_\Omega, |U| \ge C_f,$$

where $C_f \ge 0$, $\nu_0 > 0$.

C2)

$$\|D_u D_u G(x,y,U)\| \le C_f(1+|U|^{p_2}), \qquad \forall |(x,y)| \le C_\Omega, U \in \mathbf{R}^d,$$

$$\|D_{(x,y)} D_u G(x,y,U)\| \le C_f(1+|U|^{p_2+1}), \qquad \forall |(x,y)| \le C_\Omega, U \in \mathbf{R}^d,$$

where p_2, $C_f \ge 0$. If $M+N > 2$, then additionally $p_2 < 2/(M+N-2)$.

C3)

$$\sum_{i,l=1}^{d} d_i \xi_i \partial^2_{u_l,u_i} G(x,y,U) \xi_l \ge -C_f |\xi|^2, \qquad \forall |(x,y)| \le C_\Omega, U, \xi \in \mathbf{R}^d,$$

where $C_f \ge 0$, and d_i are the entries of the diagonal matrix T of equation (1.3).

We want to make some comments on the conditions on f and G.

Remark 1.1. 1) *By* C2) *and* H1), *$\nabla_u G$ is of order $|U|^{p_2+1}$, and f of order $|U|^{p_0+1}$. Taking this into account in* H2), *we get $p_1 \le 2 + p_0 + p_2$, and thus some restrictions on p_2. We have the following four cases, depending on whether or not each of the following statements holds:*

i) *f explicitly depends on* (x, y).

ii) *G explicitly depends on* (x, y) *or* $\hat{B}_\varepsilon \neq 0$.

If i) *and* ii) *are true, then necessarily* $p_2 = p_0$, *and thus* $p_1 = 2(p_0 + 1)$.
If i) *is true but* ii) *false, then* $p_2 \geq p_0$, *and thus* $2(p_0 + 1) \leq p_1 \leq p_0 + p_2 + 2$.
If i) *is false and* ii) *true, then* $p_2 \leq p_0$, *and thus* $2(p_2 + 1) \leq p_1 \leq p_0 + p_2 + 2$.
If both i) *and* ii) *are false, then the only restriction on* p_2 *is* $p_0 + p_2 > 0$.

2) *If we choose* $G(U) = |U|^2/2$, *then* G *satisfies conditions* C1), C2), C3). *Because of the remark above, in this case* $f = f(U)$ *has to be independent of* (x, y), *and* H2) *becomes* H2′)*:*
H2′)

$$U \cdot f(U) \leq C_f - \mu_0 |U|^{p_1}, \qquad \forall |(x, y)| \leq C_\Omega, U \in \mathbf{R}^d,$$

where $p_1 > 2$, $\mu_0 > 0$, $C_f \geq 0$.
We use the more general G *than* $|U|^2/2$, *because this allows an* (x, y) *dependence on* f, *and also a sublinear growth of* f, *if* $\hat{B}_\varepsilon = 0$.
All our conclusions hold if $f = f(U)$ *satisfies* H2′) *instead of* H2).

3) *Condition* H1) *makes the Nemitsky operator* $\hat{f}_\varepsilon$ *locally Lipschitz* $(0 \leq \varepsilon \leq 1)$, *and* H2), H3) *assure the boundedness of all trajectories of the nonlinear flow* π_ε, *which leads to the existence of a global attractor. Similar conditions can be found e.g. in* [2].

The rest of this article is organized as follows.

In section 2 we prove the convergence of the linear and nonlinear semi-groups, respectively. In Section 3 we derive some general conditions on the nonlinearity which suffice for the existence of attractors of the corresponding semi-groups. They also imply the upper-semi-continuity of these attractors. In the last section we prove our main result. That is, we treat the important example that the nonlinearity is the Nemitsky operator of a map, and give sufficient conditions on this map so that the conditions of section 3 are satisfied.

2. Convergence of the semi-groups

In this section we investigate the abstract equation

$$U_t = -A_\varepsilon U + \hat{F}_\varepsilon(U), \qquad t > 0 \tag{2.1}$$

as $\varepsilon \downarrow 0$ (e.g. equation (1.6), but here we do not suppose $\hat{F}_\varepsilon$ to be a Nemitsky operator).

We closely follow the ideas of [9].

First note that the bilinear forms

$$\alpha_\varepsilon : H^1 \times H^1 \to \mathbf{R}, \qquad \alpha_\varepsilon(u,v) = \int_\Omega \nabla_\varepsilon u \nabla_\varepsilon v \, dxdy,$$

$$\alpha_0 : H_s^1 \times H_s^1 \to \mathbf{R}, \qquad \alpha_0(u,v) = \int_\Omega \nabla_x u \nabla_x v \, dxdy,$$

generate selfadjoint operators with compact resolvents

$$\beta_\varepsilon : D(\beta_\varepsilon) \subset H^1 \to L^2, \qquad \beta_0 : D(\beta_0) \subset H_s^1 \to L_s^2,$$

respectively, and we have $U = (u_1, \dots, u_d) \in D(A_\varepsilon)$ $(U \in D(A_0))$ iff $u_i \in D(\beta_\varepsilon)$ $(u_i \in D(\beta_0))$ for all $i = 1, \dots, d$. For the respective case we have

$$A_\varepsilon U = \begin{pmatrix} d_1 \beta_\varepsilon u_1 \\ \vdots \\ d_d \beta_\varepsilon u_d \end{pmatrix}, \qquad A_0 U = \begin{pmatrix} d_1 \beta_0 u_1 \\ \vdots \\ d_d \beta_0 u_d \end{pmatrix}. \tag{2.2}$$

Let λ_j^ε, λ_j^0, U_j^ε, U_j^0 denote the eigenvalues and eigenvectors of A_ε and A_0, where—unless stated otherwise—the eigenvalues are ordered increasingly, and $(U_j^\varepsilon)_{j\geq 1}$, $(U_j^0)_{j\geq 1}$ are orthonormal systems (ONS) of $(L^2)^d$ and $(L_s^2)^d$, respectively.

We assume that there exists a set $S \subset (H^1)^d$ such that the operator $\hat{F}_\varepsilon(U) : S \to (L^2)^d$ is Lipschitz, i.e. there is an $L \geq 0$ (independent of $\varepsilon \in [0,1]$), such that

$$|\hat{F}_\varepsilon(U) - \hat{F}_\varepsilon(V)|_{L^2} \leq L|U - V|_{H^1}, \qquad \forall U, V \in S.$$

For a limiting semi-flow to exist, assume additionally

$$\hat{F}_0(S \cap (H_s^1)^d) \subset (L_s^2)^d,$$

and

$$|\hat{F}_\varepsilon(U) - \hat{F}_0(U)|_{L^2} \to 0, \qquad \varepsilon \downarrow 0, \tag{2.3}$$

for all $U \in (H_s^1)^d$ (e.g. if f satisfies H1), H2), H3) in §1).

It is well known that under above conditions on $\hat{F}_\varepsilon$, (2.1) together with the initial value $U(0) = U_0 \in S$ defines local semi flows π_ε on S.

Moreover,

$$U_t = -A_0 U + \hat{F}_0(U), \qquad t > 0$$

together with the initial value $U(0) = U_0 \in S \cap (H_s^1)^d$ defines a local semi-flow π_0 on $S \cap (H_s^1)^d$.

We claim that the semi-flows π_ε converge in a strong sense to the semi-flow π_0. The proofs are as in [9] with only minor changes. Alternatively, one can use the fact that A_ε and A_0 can be expressed by its one-dimensional-counterparts β_ε and β_0, respectively.

More in detail, one proves first the convergence of the eigenvalues λ_j^ε and eigenvectors U_j^ε of A_ε to their counterparts λ_j^0 and U_j^0 of A_0:

$$\lambda_j^0 = \lim_{\varepsilon \downarrow 0} \lambda_j^\varepsilon = \sup_{\varepsilon > 0} \lambda_j^\varepsilon, \qquad j = 1, 2, \ldots,$$

and for any sequence $\varepsilon_n \downarrow 0$, there is a subsequence, called (ε_n) again, such that for all $j \geq 1$ there is an $(L_s^2)^d$-complete ONS $(U_j)_{j \geq 1}$, with

$$|U_j^{\varepsilon_n} - U_j|_{\varepsilon_n} \to 0 \qquad (n \to \infty).$$

Having established the convergence of eigenvalues and eigenvectors, one proves the convergence of the linear semi-groups e^{-tA_ε} to e^{-tA_0}:

Theorem 2.1. *Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive numbers tending to* 0, $\beta > 0$, $(U_n)_{n \geq 1} \subset (L^2)^d$, $U_0 \in (L_s^2)^d$, *and* $U_n \to U_0$ *in* $|.|_{L^2}$.
Then

$$\sup_{t \in [\beta, \infty[} |e^{-A_{\varepsilon_n} t} U_n - e^{-A_0 t} U_0|_{\varepsilon_n} \to 0, \qquad n \to \infty.$$

With the convergence of the linear semi-groups, one proves the convergence of the nonlinear semi-flows π_ε to π_0:

Theorem 2.2. *Let $(\varepsilon_n)_{n \geq 1}$ be a sequence of positive numbers tending to* 0, $0 < \beta < \infty$, $U_n \in S \subset (H^1)^d$, *and* $U_0 \in S \cap (H_s^1)^d$, *where* $U_n \to U_0$ *in* $|.|_{L^2}$.
Assume $U_n \pi_{\varepsilon_n} t$, $U_0 \pi_0 t$ are defined for all $n \geq 1$ and $0 \leq t \leq \beta$.
Then for all $t_0 \in]0, \beta]$, $t_n \in [0, \beta]$ with $t_n \to t_0$, we have

$$|U_n \pi_{\varepsilon_n} t_n - U_0 \pi_0 t_0|_{\varepsilon_n} \to 0, \qquad n \to \infty.$$

We comment on the conditions in the last theorem.

The assumption in Theorem 2.2 that $U_0 \pi_0 t$ exists for $0 \leq t \leq \beta$ is unnecessary, if $\hat{F}_\varepsilon : S \subset (H^1)^d \to (L^2)^d$, and $(H_s^1)^d \subset S$. Therefore, in this case, the theorem is true under the remaining conditions.

We shall briefly outline why this assumption is not necessary.

$U_0 \pi_0 t$ exists for $0 \leq t < \delta$, a $\delta > 0$. Assume $\delta \leq \beta$ to be maximal. By Theorem 2.2 (as stated above, with $\beta = \delta$) one can without loss of generality assume $|U_n - U_0|_{\varepsilon_n} \to 0$, $U_n \in D(A_{\varepsilon_n})$, $U_0 \in D(A_0)$. It is possible to show $\sup(|U_n \pi_{\varepsilon_n} t|_{\varepsilon_n} : 0 \leq t \leq \beta, n \geq 1) < \infty$. By Theorem 2.2 this implies $\sup(|U_0 \pi_0 t| : 0 \leq t \leq \delta) < \infty$ too, and by Theorem 3.3.4 [8] $U_0 \pi_0 t$ is extendable.

As a second remark we note that Corollary 5.2 in [9] is also true for systems, i.e. a sequence of uniformly bounded solutions of π_{ε_n}, $\varepsilon_n \downarrow 0$, has a subsequence converging in $|.|_\varepsilon$ to a solution of π_0.

3. Semi continuity of attractors

In this section we shall show that if the nonlinearity $\hat{F}_\varepsilon$ in equation (2.1) satisfy some natural conditions (see A1) to A4) below), then the resulting semi-flows π_ε have global attractors $\mathscr{A}_\varepsilon$ $(0 \le \varepsilon \le 1)$, and the (a priori local) semi-flow π_0 is a global one. Also we prove the family of attractors $\mathscr{A}_\varepsilon$ to be upper-semi-continuous at $\varepsilon = 0$.

Note that in this section we do not suppose $\hat{F}_\varepsilon$ to be a Nemitsky operator.

Let $U(t)$ and π_ε denote respectively the solution and the resulting (local) semi-flow generated by the equation (2.1) with the initial condition $U(0) = U_0$, $U_0 \in (H^1)^d$, if $0 < \varepsilon \le 1$, $U_0 \in (H^1_s)^d$, if $\varepsilon = 0$. Here $\hat{F}_\varepsilon : (H^1)^d \to (L^2)^d$ is a nonlinear function.

We impose the following conditions on the nonlinearity $\hat{F}_\varepsilon$:

A1) $\hat{F}_\varepsilon$ is (locally) Lipschitz, i.e. for every $\delta > 0$ there is an $L = L(\delta)$ (independent of ε), such that for all $0 \le \varepsilon \le 1$

$$|\hat{F}_\varepsilon(U) - \hat{F}_\varepsilon(V)|_{L^2} \le L|U - V|_{H^1}, \qquad \forall U, V \in (H^1)^d, |U|_{H^1}, |V|_{H^1} \le \delta.$$

A2) For $0 < \varepsilon \le 1$ the semi-flows π_ε exist for all times $t \ge 0$. For every $\delta > 0$ there is a $C = C(\delta) > 0$ (independent of ε), such that

$$|U_0 \pi_\varepsilon t|_\varepsilon = |U(t)|_\varepsilon \le C, \qquad \forall U_0 \in (H^1)^d, |U_0|_\varepsilon \le \delta.$$

A3) The semi-flows π_ε, $0 < \varepsilon \le 1$, have absorbing sets which are bounded uniformly with respect to $|.|_\varepsilon$, i.e. there are a $\delta_f > 0$, and for every $\delta > 0$ a $T = T(\delta) > 0$, both δ_f and T being independent of ε, such that

$$|U_0 \pi_\varepsilon t|_\varepsilon < \delta_f, \qquad \forall U_0 \in (H^1)^d, |U_0|_\varepsilon \le \delta, t \ge T. \tag{3.1}$$

A4)

$$\hat{F}_0|_{(H^1_s)^d} : (H^1_s)^d \to (L^2_s)^d,$$

and $\hat{F}_\varepsilon$ approaches $\hat{F}_0$ pointwise, i.e.

$$\lim_{\varepsilon \downarrow 0} |\hat{F}_\varepsilon(U) - \hat{F}_0(U)|_{L^2} = 0, \qquad \forall U \in (H^1_s)^d.$$

Note that we do not suppose that the semi-flow π_0 exists for all $t \ge 0$. We only assume that the semi-flows π_ε $(0 < \varepsilon)$ are global.

Note also that if A1) holds, then $\hat{F}_\varepsilon$ maps bounded sets (of $(H^1)^d$ if $0 < \varepsilon$, of $(H^1_s)^d$ if $\varepsilon = 0$) into bounded sets (of $(L^2)^d$ if $0 < \varepsilon$, and of $(L^2_s)^d$ if $\varepsilon = 0$, respectively).

Roughly speaking, conditions A1) to A4) will be used in the following way.

Conditions A1), A2), A3) are sufficient for the semi-flows π_ε, $0 < \varepsilon$, to have global attractors $\mathscr{A}_\varepsilon$. These attractors are bounded uniformly in $|.|_\varepsilon$. By A2) we can change $\hat{F}_\varepsilon$ outside a certain ball in $(H^1)^d$, so that with A1) the resulting nonlinearity is globally Lipschitz, and with A4) we can apply the results of §2. That is to say, the semi-flows π_ε converge to the limit semi-flow π_0. Thus π_0 exists for all $t \geq 0$, and the absorbing sets of A3) extend to an absorbing set for π_0. Again using that by A1) $\hat{F}_\varepsilon$ maps bounded sets into bounded sets, there is a global attractor $\mathscr{A}_0$ for π_0 too. Since the attractors $\mathscr{A}_\varepsilon$, $0 < \varepsilon \leq 1$, are uniformly bounded, and the semi-flows π_ε converge to π_0, the family of attractors can be shown to be upper-semi-continuous at $\varepsilon = 0$.

We start by proving the existence of attractors for the semi-flows π_ε, $0 < \varepsilon$.

Theorem 3.1. *Let $\hat{F}_\varepsilon$ satisfy* A1), A2), A3). *Then π_ε has a global attractor $\mathscr{A}_\varepsilon$ which attracts bounded sets of $(H^1)^d$. $\mathscr{A}_\varepsilon$ is compact and connected in $(H^1)^d$. Moreover, $\mathscr{A}_\varepsilon$ is an ω-limit set*

$$\mathscr{A}_\varepsilon = \omega(\{U \in (H^1)^d : |U|_\varepsilon \leq \delta_f\}) \subset \{U \in (H^1)^d : |U|_\varepsilon \leq \delta_f\}.$$

Here δ_f is as in A3).

Proof. The proof is a simple adaptation of the proof of Theorem 3.3.6 [8], followed by Theorem 1.1, chapter 1 of [12].

By A3)

$$B^\varepsilon_{\delta_f} := \{U \in (H^1)^d : |U|_\varepsilon \leq \delta_f\}$$

attracts bounded sets of $(H^1)^d$.

We claim that for any $\delta > 0$, $\bigcup_{t>1} B^\varepsilon_\delta \pi_\varepsilon t \subset (H^1)^d$ is in a compact set.

Write equation (2.1) as follows:

$$U_t = -(A_\varepsilon + id)U + (\hat{F}_\varepsilon(U) + U) = -\tilde{A}_\varepsilon U + \tilde{F}_\varepsilon(U).$$

Since A_ε has compact resolvent, if X^α $(0 \leq \alpha)$ denotes the fractional power space of A_ε (or equivalently of $\tilde{A}_\varepsilon$), with norm $\|U\|_\alpha = |\tilde{A}_\varepsilon^\alpha U|_{L^2}$, the embedding $X^\beta \subset (H^1)^d = X^{1/2}$ is compact for $\frac{1}{2} < \beta < 1$.

So we only have to show that for all $\delta > 0$, and fixed $\frac{1}{2} < \beta < 1$, the set $\{B^\varepsilon_\delta \pi_\varepsilon t : t > 1\}$ is bounded in X^β.

Let $U_0 \in (H^1)^d$, $|U_0|_\varepsilon \leq \delta$. By A2) $|U_0 \pi_\varepsilon t|_\varepsilon \leq C(\delta)$, for all $t \geq 0$, by A1) $\hat{F}_\varepsilon$—and thus $\tilde{F}_\varepsilon$ too—maps bounded sets of $(H^1)^d$ into bounded sets of $(L^2)^d$, thus there is a constant C_1, such that $|\tilde{F}(U_0 \pi_\varepsilon t)|_{L^2} \leq C_1$, for all $t \geq 0$.

Since $Re\, \sigma(\tilde{A}_\varepsilon) > \frac{1}{2}$, we get for $t \geq 1$

$$\begin{aligned}\|U(t)\|_\beta &\leq \|e^{-\tilde{A}_\varepsilon t} U_0\|_\beta + \int_0^t \|e^{-\tilde{A}_\varepsilon (t-s)} \tilde{F}_\varepsilon(U(s))\|_\beta ds \\ &\leq C_2 \left(t^{-\beta} e^{-(1/2)t} \delta + C_1 \left(\frac{1}{1-\beta} + 2(e^{-1/2} - e^{-(1/2)t}) \right) \right),\end{aligned}$$

where C_2 is a constant independent of U_0 and t. This proves the claim.

Note for later use that for the proof of the claim we only needed A_ε to have compact resolvent, $\|U(t)\|_{1/2}$ to be bounded uniformly on bounded sets of $X^{1/2}$, and $\hat{F}_\varepsilon$ to map bounded sets of $X^{1/2}$ into bounded sets of X.

With the claim, by Theorem 1.1, chapter 1 of [12], the $\omega-$limit set of $B^\varepsilon_{\delta_f} \subset (H^1)^d$ with respect to π_ε,

$$\mathscr{A}_\varepsilon := \omega_\varepsilon(B^\varepsilon_{\delta_f}),$$

is a global, compact, connected attractor of bounded sets.

Now we prove the counterpart of Theorem 3.1 for the limiting semi-flow π_0. Note that we have not supposed π_0 to exist for all t, nor the existence of an absorbing set.

Theorem 3.2. *Let $\hat{F}_\varepsilon$ satisfy* A1) *to* A4). *Denote by $\hat{F}_s$ the restriction of $\hat{F}_0$ to $(H^1_s)^d$.*

Then the equation

$$U_t = -A_0 U + \hat{F}_s(U), \qquad t > 0, \tag{3.2}$$

with initial condition $U(0) = U_0 \in (H^1_s)^d$, defines via $U_0\pi_0 t := U(t)$ a global semi-flow π_0 on $(H^1_s)^d$.

π_0 has a global attractor $\mathscr{A}_0$ which attracts bounded sets of $(H^1_s)^d$. $\mathscr{A}_0$ is compact and connected in $(H^1_s)^d$. Moreover, $\mathscr{A}_0$ is the ω-limit set with respect to π_0

$$\mathscr{A}_0 = \omega(\{U \in (H^1_s)^d : |U|_{H^1} \le \tilde{\delta}_f\}) \subset \{U \in (H^1_s)^d : |U|_{H^1} \le \tilde{\delta}_f\},$$

where $\tilde{\delta}_f = (\min(1, d_1, \ldots, d_d))^{-1/2}\delta_f$, δ_f as in A3).

Proof. By A1) and A4) $\hat{F}_s$ satisfies (on bounded sets) all the conditions we posed in §2, so (3.2) defines a (a priori local) semi-flow π_0 on $(H^1_s)^d$.

Assume the solution of (3.2) with initial value $U_0 \in (H^1_s)^d$ exists for $0 \le t < T_1(U_0)$.

Denote by $V_\varepsilon(t) = U_0\pi_\varepsilon t$ the solution of equation (2.1) with initial value $U_0 \in (H^1_s)^d$, $0 < \varepsilon \le 1$.

By A2) there is a constant C, independent of ε, U_0 and t, but depending on $|U_0|_{H^1}$, such that

$$|V_\varepsilon(t)|_\varepsilon \le C, \qquad \forall 0 \le t < T_1(U_0), 0 < \varepsilon \le 1.$$

We apply Theorem 2.2. Then for all $0 < t_0 < T_1(U_0)$

$$|U(t_0)|_{H^1} \le (\min(1, d_1, \ldots, d_d))^{-1/2} C, \tag{3.3}$$

i.e. if $T_1(U_0) < \infty$, then $U(t)$ remains bounded in $(H^1_s)^d$, as $t \uparrow T_1(U_0)$.

By A1) $\hat{F}_s$ maps bounded sets into bounded sets, and by Theorem 3.3.4 [8], $T_1(U_0) = \infty$ follows. Hence π_0 is a global semi-flow.

Note also that for every $\delta > 0$ there is a $C = C(\delta) > 0$, such that $|U_0\pi_0 t|_{H^1} < C$, for all $|U_0|_{H^1} \le \delta$, $t \ge 0$. This allows us to use the results of §2, although $\hat{F}_\varepsilon$ $(0 \le \varepsilon \le 1)$ may not be globally Lipschitz.

Now take any sequence $\varepsilon_n \downarrow 0$. Let δ_f be as in A3). Then for all $\delta > 0$ there is a $T(\delta) > 0$, such that for all $n \ge 1$, $U_0 \in (H_s^1)^d$, $|U_0|_{\varepsilon_n} \le (\max(1, d_1, \ldots, d_d))^{1/2}$, $|U_0|_{H^1} < \delta$ implies $|V_{\varepsilon_n}(t)|_{\varepsilon_n} < \delta_f$, and thus applying Theorem 2.2 we get

$$|U(t)|_{H^1} \le (\min(1, d_1, \ldots, d_d))^{-1/2}\delta_f = \tilde{\delta}_f, \qquad t \ge T(\delta).$$

This means, setting for $\delta > 0$

$$B_\delta^0 := \{U \in (H_s^1)^d : |U|_{H^1} \le \delta\},$$

that $B_{\tilde{\delta}_f}^0$ absorbs bounded sets of $(H_s^1)^d$.

Since A_0 has compact resolvent, with (3.3) and A1) we can use the same argument as in the proof of Theorem 3.1 to show that $\bigcup_{t>1} B_\delta^0 \pi_0 t$ is in a compact set for any $\delta > 0$. Thus, again as in the proof of Theorem 3.1, the ω-limit set of the semi-flow π_0,

$$\mathscr{A}_0 := \omega(B_{\tilde{\delta}_f}^0),$$

is a global, compact, connected attractor of bounded sets.

We are now able to prove the main result of this section.

Theorem 3.3. *Let $\hat{F}_\varepsilon$ satisfy* A1) *to* A4)*, and $\mathscr{A}_\varepsilon$, $0 \le \varepsilon \le 1$, be the global attractors of the semi-flows π_ε of equations* (2.1) *and* (3.2)*, respectively.*

Then the family $\mathscr{A}_\varepsilon$ is upper-semi-continuous at $\varepsilon = 0$ with respect to the family of norms $|.|_\varepsilon$, i.e.

$$\lim_{\varepsilon \downarrow 0} \sup_{U \in \mathscr{A}_\varepsilon} \inf_{V \in \mathscr{A}_0} |U - V|_\varepsilon = 0.$$

Proof. In the following we write $|.|_n$, π_n, $\mathscr{A}_n$ for $|.|_{\varepsilon_n}$, π_{ε_n}, $\mathscr{A}_{\varepsilon_n}$, respectively.

Set

$$S := \{U \in (H^1)^d : \exists \text{ sequence } \varepsilon_n \downarrow 0, \mathscr{A}_n \ni V_n \rightharpoonup U \text{ in } (H^1)^d\}.$$

By Theorem 3.1, such a sequence $\{V_n\}$ is bounded, $|V_n|_n \le \delta_f$, and there is a subsequence converging weakly in $(H^1)^d$ to an element in $(H_s^1)^d$, i.e. $S \subset (H_s^1)^d$.

We claim that S is π_0-invariant.

To prove this, let $t_1 > 0$ and $U \in S$.

There are sequences $\varepsilon_n \downarrow 0$, and $U_n \in \mathscr{A}_n$ such that $U_n \rightharpoonup U$ in $(H^1)^d$. Thus $U_n \to U$ in L^2. Since $\mathscr{A}_\varepsilon$ is π_ε-invariant, there is a $V_n \in \mathscr{A}_n$, such that

$$V_n\pi_n t_1 = U_n.$$

By Theorem 3.1, $|V_n|_n \le \delta_f$, and there is a subsequence, called (V_n) again, which converges weakly in $(H^1)^d$, and strongly in $(L^2)^d$, to an element $V \in S$.

By Theorem 3.2, $V\pi_0 t$ exists for all $t \ge 0$, and with Theorem 2.2, for all $\frac{1}{2}t_1 \le t \le t_1$,

$$|V_n\pi_n t - V\pi_0 t|_n \to 0, \qquad n \to \infty. \tag{3.4}$$

Choosing $t = t_1$, we get $U = V\pi_0 t_1$. Choosing $t = \frac{1}{2}t_1$, we see $V\pi_0 \frac{t_1}{2} \in S$, and

$$U = \left(V\pi_0 \frac{t_1}{2}\right)\pi_0 \frac{t_1}{2} \in S\pi_0 \frac{t_1}{2}.$$

That is, $S \subset S\pi_0 \frac{t_1}{2}$, and S is negatively invariant with respect to π_0.

Analogously

$$|U_n\pi_n t_1 - U\pi_0 t_1|_n \to 0, \qquad n \to \infty,$$

implying $U\pi_0 t_1 \in S$. Thus $S\pi_0 t_1 \subset S$, i.e. S is positively invariant too. This proves the claim.

S is not only π_0-invariant, but $S \subset \mathscr{A}_0$. Indeed, if $U \in S$, $\mathscr{A}_n \ni V_n \rightharpoonup U$ in $(H^1)^d$, then with Theorem 3.1 and $\tilde{\delta}_f$ as in Theorem 3.2

$$|U|_{H^1} \le \liminf_{n\to\infty} |V_n|_{H^1} \le \liminf_{n\to\infty} (\min(1, d_1, \ldots, d_d))^{-1/2} |V_n|_n \le \tilde{\delta}_f.$$

Using the characterization of $\mathscr{A}_0$ of Theorem 3.2, the invariance of S implies $S \subset \mathscr{A}_0$.

Now we are able to prove the conclusion of Theorem 3.3.

Assume it to be false. Then there are a sequence $(\varepsilon_n)_{n\ge 1}$ of positive numbers tending to 0, $\delta > 0$, and $U_n \in \mathscr{A}_n$ such that for all $U_0 \in \mathscr{A}_0$,

$$|U_n - U_0|_n > \delta. \tag{3.5}$$

As before, by Theorem 3.1, $|U_n|_n \le \delta_f$, and taking a subsequence we can without loss of generality assume that

$$U_n \rightharpoonup U \in (H^1)^d \qquad \text{in } (H^1)^d.$$

But then $U \in S$.

Arguing as in the proof of the claim above, for a given $t_1 > 0$, letting $V_n \in \mathscr{A}_n$ be such that $V_n\pi_n t_1 = U_n$, and taking $V \in S \subset \mathscr{A}_0$ as a weak limit (of a subsequence) of V_n, by equation (3.4) we see

$$|U_n - V\pi_0 t_1|_n \to 0, \qquad n \to \infty.$$

By the comments above, $V\pi_0 t_1 \in S \subset \mathscr{A}_0$, which contradicts (3.5).

4. Special cases

In this section we shall treat the special case in which the function $\hat{F}_\varepsilon$ of the last section is the Nemitsky operator of a (nonlinear) map f plus a linear map of the x-derivatives of U. We shall show that if the nonlinearity f satisfies some natural conditions (i.e. H1), H2), H3)), then the Nemitsky operator $\hat{F}_\varepsilon$ satisfies conditions A1) to A4) in §3. Thus we can apply the general results of that section, and the semi-flows generated by equation (1.6), i.e. by

$$U_t = -A_\varepsilon U + \hat{B}_\varepsilon U + \hat{f}_\varepsilon(U) = -A_\varepsilon U + \hat{F}_\varepsilon(U), \qquad t > 0. \tag{4.1}$$

will be global, and have attractors $\mathscr{A}_\varepsilon$ which are upper-semi-continuous at $\varepsilon = 0$.

We make the same assumptions as in §1, i.e. we suppose

$$\hat{B}_0|_{(H_s^1)^d} : (H_s^1)^d \to (L_s^2)^d,$$

and G and f to satisfy conditions C1), C2), C3) and H1), H2), H3), respectively. Note that $\hat{B}_\varepsilon : (H^1)^d \to (L^2)^d$ is bounded uniformly in ε, $(0 \le \varepsilon \le 1)$.

We want to apply the results of §3. For this we have to prove $\hat{F}_\varepsilon$ to satisfy A1) to A4). These proofs are rather technical and long, so let us first state the results and present the proofs afterwards.

Lemma 4.1. *Let $f \in C^1(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}^d)$ satisfy* H1). *Let $\hat{F}_\varepsilon$ be as defined in* (1.9) *and assume $\hat{B}_0|_{(H_s^1)^d} : (H_s^1)^d \to (L_s^2)^d$.*

Then $\hat{F}_\varepsilon$ satisfies conditions A1) *and* A4) *in* §3, *for $0 \le \varepsilon \le 1$.*

Theorem 4.1. *Let G and f satisfy conditions* C1), C2), C3), *and* H1), H2), H3), *respectively. Define the operators $\hat{B}_\varepsilon$, $\hat{f}_\varepsilon$, $\hat{F}_\varepsilon$ as in* (1.7), (1.8), *and* (1.9), *respectively. Then the solution $U(t)$ to equation* (4.1) *with initial value $U(0) = U_0 \in (H^1)^d$ is uniquely defined and exists for all $t \ge 0$.*

Moreover, there is an $\delta_f > 0$ such that for every $0 < \delta$ there is a $T = T(\delta) > 0$, both δ_f and T independent of ε, and

$$|U(t)|_\varepsilon < \delta_f, \qquad \forall |U_0|_\varepsilon < \delta, t \ge T.$$

Also,

$$|U(t)|_\varepsilon^2 \le (1 + |U_0|_\varepsilon^{2+p_2})\delta_f, \qquad \forall U_0 \in (H^1)^d, t \ge 0.$$

Theorem 1.1, the main result of this article, is now a simple corollary, using Lemma 4.1, Theorem 4.1, and the results of §3.

We have to prove Lemma 4.1 and Theorem 4.1. The easy part is Lemma 4.1. To prove it, we proceed through three lemmas stating some facts about Nemitsky operators.

Lemma 4.2. *Let* $g \in C^1(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R})$, *and assume* $\sup(|\nabla g(x,y,s)| : (x,y) \in \bar{\Omega}, s \in \mathbf{R}^d) < M_g \in \mathbf{R}$.

If $U = (u_1, \dots, u_d) \in (H^1)^d$, *then* $\hat{g}U(x,y) := g(x,y,U(x,y)) \in H^1$, *and the derivatives of* $\hat{g}$ *are computed according to the usual chain rule.*

Lemma 4.2 can be proven by slightly modifying the proof of Proposition IX.5 in [3]. Note that Ω is bounded, so the condition $G(0) = 0$ in [3] is not needed here.

As in Theorem 5.3 of [9] we have

Lemma 4.3. *If* $g \in C^1(\mathbf{R}^M \times \mathbf{R}^d, \mathbf{R})$, $U \in (H_s^1)^d$, *and*

$$\hat{g}U(x,y) := g(x, U(x,y)) \in L^2,$$

then $\hat{g}U \in L_s^2$.

It is a standard procedure to prove $\hat{f}_\varepsilon$ to be well defined and locally Lipschitz. Using H1), the Sobolev-Imbedding-Theorem, and Lemma 4.3, we find

Lemma 4.4. *Let* $f \in C^1(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}^d)$ *satisfy* H1). *Then the Nemitsky operator* $\hat{f}_\varepsilon : (H^1)^d \to (L^2)^d$, *defined in* (1.8) *is well defined and locally Lipschitz. More precisely, for* $\varepsilon \in [0,1]$, $U, V \in (H^1)^d$, $|U|_{H^1}, |V|_{H^1} \le \delta$, *we have*

$$|\hat{f}_\varepsilon(U) - \hat{f}_\varepsilon(V)|_{L^2} \le C(1 + \delta^{p_0})|U - V|_{H^1}, \tag{4.2}$$

where $C > 0$ *is a constant, independent of* U, V, δ *and* ε.

If $U \in (H_s^1)^d$, *then*

$$|\hat{f}_\varepsilon(U) - \hat{f}_0(U)|_{L^2} \le \varepsilon C(1 + |U|_{H^1}^{p_0+1}). \tag{4.3}$$

The restriction of $\hat{f}_0$ *to* $(H_s^1)^d$ *satisfies*

$$\hat{f}_0|_{(H_s^1)^d} : (H_s^1)^d \to (L_s^2)^d. \tag{4.4}$$

Lemma 4.1 is now an easy consequence of Lemma 4.4, the continuity of B^j, and the boundedness of $\hat{B}_\varepsilon$.

We proceed to the proof of Theorem 4.1. Again we use a number of lemmas: Lemma 4.5 gives an approximation in L^∞ of an eigenvector of β_ε, which is used in Lemma 4.6 to provide an approximation $V \in (L^\infty)^d$ of an $U \in D(A_\varepsilon)$, which in turn is needed in Lemma 4.10. Lemma 4.7 states an estimate needed in Lemma 4.8, which collects several useful facts about G, or rather about the Nemitsky operators $\hat{G}$ and $\widehat{\nabla_u G}$. Each of the Lemmas 4.9, 4.10 and 4.11 provides an upper bound for part of an expression which arises in the proof of Theorem 4.1.

Lemma 4.5. *Let $u \in H^1$ be an eigenvector of β_ε, and $\delta > 0$.*
Then there is a $v \in D(\beta_\varepsilon) \cap L^\infty$, such that

$$|u - v|_{H^1} < \delta, \qquad |\beta_\varepsilon(u - v)|_{L^2} < \delta.$$

Proof. Let u be an eigenvector of β_ε, and λ the corresponding eigenvalue. Without loss of generality, assume u to be normalized in L^2.

If $\lambda = 0$, then without loss of generality $u \equiv$ constant, and $v = u$ satisfies the conclusion.

For the rest of the proof assume $\lambda > 0$.

Let $(u_j)_{j \geq 1}$ be an L^2-ONS of eigenvectors of β_ε, with corresponding eigenvalues $(\lambda_j)_{j \geq 1}$ (see [9]). Without loss of generality, assume $\lambda_1 = 0$, $u_1 = |\Omega|^{-1/2}$, $\lambda_2 = \lambda$, and $u_2 = u$.

Let $\delta_1 > 0$, and $|v|_\varepsilon^2 := |v|_{L^2}^2 + \alpha_\varepsilon(v, v)_{L^2}$, for all $v \in H^1$.

There is a $\tilde{v} \in C_0^\infty(\mathbf{R}^{M+N})$ with $|\tilde{v} - u|_{H^1} < \delta_1$ (see e.g. [1], Lemma A 5.8), and a $v \in D(\beta_\varepsilon) \cap L^\infty$, such that

$$(\beta_\varepsilon + id)v = (\lambda + 1)\tilde{v}.$$

Hence

$$\begin{aligned}
|v|_{L^2}^2 &\leq |v|_\varepsilon^2 \leq (\lambda + 1)|\tilde{v}|_{L^2}|v|_{L^2} \leq (\lambda + 1)(\delta_1 + |u|_{H^1})|v|_{L^2},\\
|u - v|_{H^1}^2 &\leq |u - v|_\varepsilon^2 = ((\beta_\varepsilon + id)(u - v), u - v)_{L^2}\\
&\leq (\lambda + 1)|u - \tilde{v}|_{L^2}(|u|_{L^2} + |v|_{L^2})\\
&\leq (\lambda + 1)\delta_1(1 + (\lambda + 1)(\delta_1 + |u|_{H^1})),
\end{aligned}$$

and

$$|u - v|_{H^1}^2 \leq C\delta_1,$$

for a suitable constant C. So

$$|\beta_\varepsilon(u - v)|_{L^2} = |(\lambda + 1)(u - \tilde{v}) - u + v|_{L^2} \leq (\lambda + 1)\delta_1 + \sqrt{C}\sqrt{\delta_1},$$

which proves the lemma.

Lemma 4.6. *Let $U \in D(A_\varepsilon)$ and $\delta > 0$.*
Then there exists a $V \in D(A_\varepsilon) \cap (L^\infty)^d$, such that

$$|U - V|_\varepsilon \leq \delta, \qquad |A_\varepsilon(U - V)|_{L^2} \leq \delta. \tag{4.5}$$

Proof. Since $U = (u_1, \ldots, u_d) \in D(A_\varepsilon)$ iff $u_j \in D(\beta_\varepsilon)$ for $j = 1, \ldots, d$, Lemma 4.6 easily follows from Lemma 4.5.

A simple estimate proves the following lemma.

LEMMA 4.7. *Let* $p \geq 1$, $u \in L^p$, *and* $C > 0$.
If $|u|_{L^p}^p \geq 2C^p|\Omega|$, *then*

$$\int_{\{(x,y)\in\Omega:|u(x,y)|\geq C\}} |u(x,y)|^p dxdy \geq \frac{1}{2}|u|_{L^p}^p.$$

LEMMA 4.8. *Let* $G \in C^2(\mathbf{R}^M \times \mathbf{R}^N \times \mathbf{R}^d, \mathbf{R}_{\geq})$ *satisfy* C1), C2). *Define the Nemitsky operators* $\hat{G}$ *and* $\widehat{\nabla_u G}$ *through*

$$\hat{G}(U)(x,y) = G(x,y,U(x,y)),$$
$$\widehat{\nabla_u G}(U)(x,y) = \begin{pmatrix} \partial_{u_1} G(x,y,U(x,y)) \\ \vdots \\ \partial_{u_d} G(x,y,U(x,y)) \end{pmatrix}.$$

Then the following hold:

i) $\widehat{\nabla_u G} : (H^1)^d \to (L^2)^d$ *is locally Lipschitz. In particular, if* $U, V \in (H^1)^d$, $|U|_{H^1}, |V|_{H^1} \leq \delta$, *then*

$$|\widehat{\nabla_u G}(U) - \widehat{\nabla_u G}(V)|_{L^2} \leq C(1+\delta^{p_2})|U-V|_{H^1},$$

where C *is a constant, independent of* U, V, δ, *and* p_2 *is as in* C2).

ii) $\hat{G} : (H^1)^d \to L^1$, *and there is a constant* $C > 0$, *such that*

$$|\hat{G}(U)|_{L^1} \leq C(1 + |U|_{H^1}^{p_2+2}), \qquad \forall U \in (H^1)^d,$$

where p_2 *is as in* C2).

iii) *There are constants* C, $\tilde{C} > 0$, *such that*

$$|U|_{L^2} \geq C \Rightarrow |\hat{G}(U)|_{L^1} \geq \tilde{C}|U|_{L^2}^2, \qquad \forall U \in (H^1)^d.$$

iv) *If* I *is an open interval,* $I \ni t \mapsto U(t) \in (H^1)^d$ *is continuous, and differentiable with respect to* $|.|_{L^2}$, *the derivative being* $U_t(t) \in L^2$, *then*

$$\frac{\partial}{\partial t}|\hat{G}(U(t))|_{L^1} = (\widehat{\nabla_u G}(U(t)), U_t(t))_{L^2}, \qquad \forall t \in I.$$

PROOF. By C2), i) follows directly from Lemma 4.4 applied to $D_u G(x,y,U)$.

Note for later use that one has $H^1 \subset L^{2(p_2+1)}$, p_2 as in C2).

A simple estimation using C2) and the Sobolev-Imbedding-Theorem proves ii).

Let $U \in (H^1)^d$, then with C1)

$$|\hat{G}(U)|_{L^1} \ge \nu_0 \int_{\{(x,y)\in\Omega:|U(x,y)|\ge C_f\}} |U|^2 dxdy.$$

Now Lemma 4.7 shows iii).

iv) is the only claim not being that simple to prove. Since

$$\begin{aligned}
&\lim_{h\to 0}\left|\frac{|\hat{G}(U(t+h)|_{L^1} - |\hat{G}(U(t)|_{L^1}}{h} - \left(\widehat{\nabla_u G}(U(t)), \frac{\partial}{\partial t}U(t)\right)_{L^2}\right| \\
&\quad \le \sum_{j=1}^{d} \lim_{h\to 0}\int_\Omega \Bigg(|\partial_{u_j}G(x,y,\xi_h(x,y))| \\
&\qquad \cdot \left|\frac{u_j(t+h)(x,y) - u_j(t)(x,y)}{h} - \frac{\partial}{\partial t}u_j(t)(x,y)\right| \\
&\qquad + |\partial_{u_j}G(x,y,\xi_h(x,y)) - \partial_{u_j}G(x,y,U(t)(x,y))| \left|\frac{\partial}{\partial t}u_j(t)(x,y)\right|\Bigg) dxdy,
\end{aligned}$$

where $\xi_h(x,y)$ is between $U(t)(x,y)$ and $U(t+h)(x,y)$, we only have to show

$$|\partial_{u_j}G(x,y,\xi_h(x,y)) - \partial_{u_j}G(x,y,U(t)(x,y))|_{L^2} \to 0, \qquad h \to 0. \tag{4.6}$$

So fix $j \in \{1,\dots,d\}$, and let $h_n \to 0$.

$t \mapsto U(t)$ is continuous in $|.|_{L^2}$, hence without loss of generality, we can assume for a.a. (x,y)

$$U(t+h_n)(x,y) \to U(t)(x,y), \qquad n \to \infty.$$

Thus for a.a. (x,y)

$$E1 := |\partial_{u_j}G(x,y,\xi_{h_n}(x,y)) - \partial_{u_j}G(x,y,U(t)(x,y))| \to 0, \qquad n \to \infty.$$

On the other hand

$$(E1)^2 \le (\|(D_u)^2 G(x,y,\eta_n(x,y))\| \cdot |\xi_{h_n}(x,y) - U(t)(x,y)|)^2,$$

for a $\eta_n(x,y)$ between $U(t)(x,y)$ and $\xi_{h_n}(x,y)$. Applying C2) it follows that

$$\begin{aligned}
(E1)^2 &\le 2C_f^2(1 + |U(t)(x,y)|^{2p_2} + |U(t+h_n)(x,y)|^{2p_2}) \\
&\qquad \cdot |U(t+h_n)(x,y) - U(t)(x,y)|^2.
\end{aligned}\tag{4.7}$$

If $p_2 = 0$, then the right-hand side goes to 0, as a function in L^1, hence by the General-Lebesgue-Convergence-Theorem (see e.g. [1], A1.23)

$$\int_\Omega |\partial_{u_j}G(x,y,\xi_{h_n}(x,y)) - \partial_{u_j}G(x,y,U(t)(x,y))|^2 dxdy \to 0, \quad n \to \infty. \tag{4.8}$$

If $p_2 > 0$, we apply the Hölder-inequality for $p = \frac{p_2+1}{p_2} > 1$ to get

$$\int_{\Omega} |U(t)(x,y)|^{2p_2} |U(t+h_n)(x,y) - U(t)(x,y)|^2 dxdy$$
$$\leq \left(\int_{\Omega} |U(t)(x,y)|^{2p_2 p} dxdy \right)^{1/p}$$
$$\cdot \left(\int_{\Omega} |U(t+h_n)(x,y) - U(t)(x,y)|^{2(p/(p-1))} dxdy \right)^{(p-1)/p}.$$

Since $2pp_2 = 2\frac{p}{p-1} = 2(p_2+1)$ and $H^1 \subset L^{2(p_2+1)}$ (see proof of i)), the continuity in $|.|_{H^1}$ of $t \mapsto U(t)$ implies that the right-hand side of the formula above tends to 0. Thus the L^1-norm of the right-hand side of (4.7) tends to 0 too. We can again apply the General-Lebesgue-Convergence-Theorem to get (4.8) also in the case $p_2 > 0$.

(4.8) immediately yields (4.6), and iv) has been proven.

The next three lemmas provide some estimates we shall need in the proof of Theorem 4.1. In all these lemmas we suppose f and G to satisfy H1), H2), H3) and C1), C2), C3), respectively.

Lemma 4.9. *For every $U \in D(A_\varepsilon)$ and $C_A > 0$ there exists a constant $C > 0$, independent of ε and U, such that*

$$(\hat{B}_\varepsilon U, T^{-1} A_\varepsilon U)_{L^2} \leq C|U|_{L^2} |A_\varepsilon U|_{L^2} + C_A |A_\varepsilon U|_{L^2}^2. \tag{4.9}$$

Proof. Let $C_1 \geq \|\hat{B}_\varepsilon\|$ be independent of ε, and set

$$\alpha_j := (U, U_j^\varepsilon)_{L^2}, \qquad d_m := \min(1, d_1, \ldots, d_d), \qquad C_2 := \frac{C_1}{d_m}.$$

Choose $j_0 \in \mathbf{N}$ such that for $x \geq \lambda_{j_0}^1$,

$$0 \leq \frac{C_A^2}{C_2^2} x^2 - x d_m^{-1} - d_m^{-1},$$

then for $j \geq j_0$, because λ_j^ε increases as ε decreases,

$$\frac{1 + \lambda_j^\varepsilon}{d_m} \leq \frac{C_A^2}{C_2^2} (\lambda_j^\varepsilon)^2,$$

and thus

$$|U|_{H^1} - \frac{C_A}{C_2} |A_\varepsilon U|_{L^2} \leq \left(\sum_{j=1}^{j_0} d_m^{-1} (1 + \lambda_j^\varepsilon) \alpha_j^2 \right)^{1/2} \leq C_3 |U|_{L^2},$$

where the constant C_3 is independent of ε and U. We get

$$(\hat{B}_\varepsilon U, T^{-1}A_\varepsilon U)_{L^2} \le (C_2C_3|U|_{L^2} + C_A|A_\varepsilon U|_{L^2})|A_\varepsilon U|_{L^2}.$$

Lemma 4.10. *Let p_0 be as in condition* H1), *p_2 as in* C2), *and $U \in D(A_\varepsilon)$. Then there exists a constant $C > 0$, independent of ε and U, such that*

$$\begin{aligned} -(\widehat{\nabla_u G}(U), A_\varepsilon U)_{L^2} + (\widehat{\nabla_u G}(U), \hat{B}_\varepsilon U)_{L^2} + (\hat{f}_\varepsilon(U), T^{-1}A_\varepsilon U)_{L^2} \\ \le C(1 + |U|_\varepsilon^2 + |U|_\varepsilon |U|_{L^{2(p+1)}}^{p+1}). \end{aligned} \tag{4.10}$$

Here p is as follows.

If $f = f(U)$ is independent of (x, y), then $p = p_2$ is independent of p_0.

If $G = G(U)$ is independent of (x, y) and $\hat{B}_\varepsilon = 0$, then $p = p_0$ is independent of p_2.

If both conditions above are satisfied, then the last term on the right-hand side in (4.10) *disappears. If neither of these conditions is satisfied, then $p = \max(p_0, p_2)$.*

Proof. We shall first show the following inequalities

$$(\hat{f}_\varepsilon(U), T^{-1}A_\varepsilon U)_{L^2} \le C(1 + |U|_\varepsilon^2 + |U|_\varepsilon |U|_{L^{2(p_0+1)}}^{p_0+1}), \tag{4.11}$$

$$-(\widehat{\nabla_u G}(U), A_\varepsilon U)_{L^2} \le C(1 + |U|_\varepsilon^2 + |U|_\varepsilon |U|_{L^{2(p_2+1)}}^{p_2+1}), \tag{4.12}$$

$$(\widehat{\nabla_u G}(U), \hat{B}_\varepsilon U)_{L^2} \le C(1 + |U|_\varepsilon^2 + |U|_\varepsilon |U|_{L^{2(p_2+1)}}^{p_2+1}). \tag{4.13}$$

For $q_1 < q_2, C > 0$, using the Hölder-inequality,

$$|U|_{L^{q_1}}^{q_1} \le C_1|U|_{L^{q_2}}^{q_1} \le C_2 + C|U|_{L^{q_2}}^{q_2}, \tag{4.14}$$

where $C_1, C_2 \ge 0$ depend only on q_1, q_2, C and Ω, so (4.11), (4.12), (4.13) imply (4.10), if the last terms in (4.11), (4.12), (4.13) disappear for f independent of (x, y), G independent of (x, y), and $\hat{B}_\varepsilon = 0$, respectively.

Note that by H1) and C2) we have in all cases $L^{2(p+1)} \subset H^1$.

By Lemmas 4.4 and 4.6 there is a sequence $U_n \in D(A_\varepsilon) \cap (L^\infty)^d$, such that

$$\begin{aligned} |U_n - U|_\varepsilon, \qquad |A_\varepsilon(U_n - U)|_{L^2}, \qquad |\hat{f}_\varepsilon(U_n) - \hat{f}_\varepsilon(U)|_{L^2}, \\ |U_n - U|_{L^{2(p_0+1)}} \to 0, \qquad n \to \infty. \end{aligned} \tag{4.15}$$

Hence it is sufficient to prove (4.11) under the assumption $U \in D(A_\varepsilon) \cap (L^\infty)^d$. In this case $\hat{f}_\varepsilon(U) \in (H^1)^d$ by Lemma 4.2.

Using H1) and H3)

$$
\begin{aligned}
&(\hat{f}_\varepsilon(U), T^{-1}A_\varepsilon U)_{L^2} \\
&\quad = \sum_{i=1}^{d} \int_\Omega \nabla_\varepsilon u_i \nabla_\varepsilon \hat{f}_{\varepsilon,i}(U(x,y))dxdy \\
&\quad = \sum_{i=1}^{d} \int_\Omega \left[\sum_{j=1}^{M} \partial_{x_j} u_i \partial_{x_j} f_i(x, \varepsilon y, U) + \frac{1}{\varepsilon} \sum_{j=1}^{N} \partial_{y_j} u_i \partial_{y_j} f_i(x, \varepsilon y, U) \right. \\
&\qquad \left. + \sum_{l=1}^{d} \left(\sum_{j=1}^{M} \partial_{x_j} u_i \partial_{u_l} f_i(x, \varepsilon y, U) \partial_{x_j} u_l + \frac{1}{\varepsilon^2} \sum_{j=1}^{N} \partial_{y_j} u_i \partial_{u_l} f_i(x, \varepsilon y, U) \partial_{y_j} u_l \right) \right] dxdy \\
&\quad \le C_3 \left(|U|_\varepsilon \left(\int_\Omega (1 + |U|^{2(p_0+1)}) dxdy \right)^{1/2} + |U|_\varepsilon^2 \right),
\end{aligned}
$$

where the constant C_3 is independent of ε and U. If f is independent of (x, y), then $(\hat{f}_\varepsilon(U), T^{-1}A_\varepsilon U)_{L^2} \le C|U|_\varepsilon^2$. (4.11) follows immediately.

Now write as

$$
-(\widehat{\nabla_u G}(U), A_\varepsilon U)_{L^2} = (-T\widehat{\nabla_u G}(U), T^{-1}A_\varepsilon U)_{L^2},
$$

and use exactly the same argument for $-T\widehat{\nabla_u G}(U)$ as for $\hat{f}_\varepsilon(U)$: Lemma 4.8 i) proves $\widehat{\nabla_u G}$—and hence $T\widehat{\nabla_u G}$ too—to be Lipschitz. Thanks to Lemma 4.6 we get the sequence corresponding to (4.15), by C2) and C3) $-T\widehat{\nabla_u G}$ satisfies H1) (with p_2 instead of p_0) and H3). This proves (4.12).

Note again, that if G is independent of (x, y), then $-(\widehat{\nabla_u G}(U), A_\varepsilon U)_{L^2} \le C|U|_\varepsilon^2$.

Now let $C_4 \ge \|\hat{B}_\varepsilon\|$ be independent of ε. Then using C2)

$$
(\widehat{\nabla_u G}(U), \hat{B}_\varepsilon U)_{L^2} \le 4C_4 |U|_{H^1} (|\widehat{\nabla_u G}(0)|_{L^2} + C_f(|U|_{L^2} + |U|_{L^{2(p_2+1)}}^{p_2+1}))
$$

which implies (4.13).

The following lemma is an easy consequence of H2).

Lemma 4.11. *There is a constant $C > 0$, independent of ε, such that for all $U \in D(A_\varepsilon)$*

$$
(\widehat{\nabla_u G}(U), \hat{f}_\varepsilon(U))_{L^2} \le C - \mu_0 |U|_{L^{p_1}}^{p_1}, \tag{4.16}
$$

where p_1 is as in H2).

Proof of Theorem 4.1. Define

$$
\mathscr{G}(\varepsilon, U_0; t) := |\hat{G}(U(t))|_{L^1} + \frac{1}{2} a_\varepsilon(U(t), T^{-1}U(t)),
$$

where $\hat{G}$ is as in Lemma 4.8, and $T = diag(d_1, \ldots, d_d)$ as before.

We shall prove $\mathscr{G}$ to be differentiable with respect to t, and $\partial_t \mathscr{G}(\varepsilon, U_0; t) \le -1$ if $\mathscr{G}(\varepsilon, U_0; t)$ is big enough. For $|U(t)|_\varepsilon$ big enough, $\mathscr{G}(\varepsilon, U_0; t)$ can be bounded from below and above by expressions in $|U(t)|_\varepsilon$. The conclusions of Theorem 4.1 then follow directly from the behavior of $\mathscr{G}(\varepsilon, U_0; t)$.

In this proof all constants $C_1, C_2, \dots$ will be independent of t, ε, U_0.

Lemma 4.1 ensures that equation (4.1) has a unique solution $U(t) \in D(A_\varepsilon) \subset (H^1)^d$, for $0 < t < T_1 = T_1(U_0)$ (see e.g. [8], Theorem 3.3.3).

Set

$$\alpha_j = \alpha_j(t) := (U(t), U_j^\varepsilon)_{L^2}, \qquad d_M := \max(d_1, \dots, d_d).$$

By Theorem 3.5.2 [8], $]0, T_1[\ni t \mapsto \frac{\partial}{\partial t} U(t)$ is continuous in $|.|_{L^2}$. Since $t \mapsto \hat{f}_\varepsilon(U(t))$ is continuous, $t \mapsto A_\varepsilon U(t)$ is continuous in $|.|_{L^2}$ too. But since

$$|U|_\varepsilon^2 = \sum_{j \ge 1} (1 + \lambda_j^\varepsilon)(U, U_j^\varepsilon)_{L^2}^2 \le 2 \sum_{j \ge 1} (1 + (\lambda_j^\varepsilon)^2)(U, U_j^\varepsilon)_{L^2}^2 = 2(|U|_{L^2}^2 + |A_\varepsilon U|_{L^2}^2),$$

$t \mapsto U(t)$ is continuous in $|.|_\varepsilon$ and $|.|_{H^1}$ too.

We have $A_\varepsilon T^{-1} = T^{-1} A_\varepsilon$, so that for $0 < t < T_1$

$$\begin{aligned}
\frac{\partial}{\partial t} a_\varepsilon(U(t), T^{-1} U(t)) &= \lim_{h \to 0} \left[\left(A_\varepsilon U(t+h), T^{-1} \frac{U(t+h) - U(t)}{h} \right)_{L^2} \right. \\
&\qquad \left. + \left(A_\varepsilon U(t), T^{-1} \frac{U(t+h) - U(t)}{h} \right)_{L^2} \right] \\
&= 2 \left(A_\varepsilon U(t), T^{-1} \frac{\partial}{\partial t} U(t) \right)_{L^2} \\
&= 2 \left(\frac{\partial}{\partial t} U(t), T^{-1} A_\varepsilon U(t) \right)_{L^2}.
\end{aligned}$$

We can apply Lemma 4.8 iv) to get for $U_0 \in (H^1)^d$ and $t > 0$

$$\begin{aligned}
\frac{\partial}{\partial t} \mathscr{G}(\varepsilon, U_0; t) &= -(\widehat{\nabla_u G}(U(t)), A_\varepsilon U(t))_{L^2} + (\widehat{\nabla_u G}(U(t)), \hat{B}_\varepsilon U(t))_{L^2} \\
&\quad + (\widehat{\nabla_u G}(U(t)), \hat{f}_\varepsilon(U(t)))_{L^2} - (A_\varepsilon U(t), T^{-1} A_\varepsilon U(t))_{L^2} \\
&\quad + ((\hat{B}_\varepsilon U(t), T^{-1} A_\varepsilon U(t))_{L^2} + (\hat{f}_\varepsilon(U(t)), T^{-1} A_\varepsilon U(t))_{L^2}. \qquad (4.17)
\end{aligned}$$

Applying Lemmas 4.9, 4.10, and 4.11 to bound each term in (4.17) above, we get

$$\frac{\partial}{\partial t}\mathscr{G}(\varepsilon, U_0; t) \leq C_1(1 + |U(t)|_\varepsilon^2 + |U(t)|_{L^2}|A_\varepsilon U(t)|_{L^2})$$
$$+ C_2|U(t)|_\varepsilon |U(t)|_{L^{2(p+1)}}^{p+1} - \frac{1}{2d_M}|A_\varepsilon U(t)|_{L^2}^2 - \mu_0|U(t)|_{L^{p_1}}^{p_1}, \quad (4.18)$$

where p is as follows:

If $f = f(U)$, $G = G(U)$ are independent of (x, y), and $\hat{B}_\varepsilon = 0$, then $C_2 = 0$, if only $f = f(U)$, then $p = p_2$, if $G = G(U)$ and $\hat{B}_\varepsilon = 0$, then $p = p_1$, and finally if f depends on (x, y), and G depends on (x, y) or $\hat{B}_\varepsilon \neq 0$, then $p = \max(p_0, p_2)$.

Note that $C_2 = 0$ or $p_1 \geq 2(p+1)$ holds by assumption. Then, as in (4.14), there is a $C_3 > 0$, such that

$$C_2|U(t)|_\varepsilon |U(t)|_{L^{2(p+1)}}^{p+1} - \frac{1}{2}\mu_0|U(t)|_{L^{p_1}}^{p_1} \leq C_3(1 + |U(t)|_\varepsilon^2). \quad (4.19)$$

Also, for any given $C > 0$ there exist constants $\tilde{C}, \overline{C} > 0$, such that

$$|U(t)|_\varepsilon^2 + |U(t)|_{L^2}|A_\varepsilon U(t)|_{L^2} \leq \tilde{C}|U(t)|_{L^2}^2 + |U(t)|_\varepsilon^2 + \frac{1}{2}C|A_\varepsilon U(t)|_{L^2}^2$$
$$= \sum_{j\geq 1} \underbrace{\left(\tilde{C} + 1 + \lambda_j^\varepsilon + \frac{1}{2}C(\lambda_j^\varepsilon)^2\right)}_{\leq \overline{C} + C(\lambda_j^\varepsilon)^2} \alpha_j^2 \leq \overline{C}|U(t)|_{L^2}^2 + C|A_\varepsilon U(t)|_{L^2}^2. \quad (4.20)$$

Inserting (4.19) and (4.20), with appropriately chosen C, and using (4.14), (4.18) becomes

$$\frac{\partial}{\partial t}\mathscr{G}(\varepsilon, U_0; t) \leq C_4(1 + |U(t)|_{L^2}^2) - \frac{1}{2}\mu_0|U(t)|_{L^{p_1}}^{p_1} - |U(t)|_\varepsilon^2 \leq C_5 - |U(t)|_\varepsilon^2. \quad (4.21)$$

Now we need a bound on $\mathscr{G}$. For later use we shall do it in both directions.

By Lemma 4.8 ii) and iii), there exist constants $C_6, C_7, C_8 > 0$, such that for all $U \in (H^1)^d$

$$C_7|U|_\varepsilon^2 \leq C_7|U|_{L^2}^2 + \frac{1}{2}a_\varepsilon(U, T^{-1}U)$$
$$\leq |\hat{G}(U)|_{L^1} + \frac{1}{2}a_\varepsilon(U, T^{-1}U), \qquad |U|_{L^2} > C_6, \quad (4.22)$$

$$|\hat{G}(U)|_{L^1} + \frac{1}{2}a_\varepsilon(U, T^{-1}U) \leq C_8(1 + |U|_\varepsilon^{p_2+2}). \quad (4.23)$$

Thus there is a $\delta_1 > 0$, independent of ε, U_0, t, such that

$$\frac{\partial}{\partial t}\mathscr{G}(\varepsilon, U_0; t) \le -1, \qquad \text{if } \mathscr{G}(\varepsilon, U_0; t) \ge \delta_1. \tag{4.24}$$

This inequality implies

$$\mathscr{G}(\varepsilon, U_0; t) \le \mathscr{G}(\varepsilon, U_0; 0) + \delta_1, \qquad 0 \le t < T_1,$$

and if $\mathscr{G}(\varepsilon, U_0; 0) = |\hat{G}(U_0)|_{L^1} + \frac{1}{2}a_\varepsilon(U_0, T^{-1}U_0) \le \delta$, then

$$\mathscr{G}(\varepsilon, U_0; t) \le 2\delta_1, \qquad T_2(\delta) \le t < T_1,$$

where $T_2(\delta)$ is independent of ε, U_0.

By (4.22) and (4.23) these inequalities imply similar ones for $|U(t)|_\varepsilon$. That is, there exist a $\delta_f > 0$ and a $T = T(\delta) > 0$ for each $\delta > 0$, both independent of ε, such that

$$\begin{aligned} &|U(t)|_\varepsilon^2 \le \delta_f(1 + |U_0|_\varepsilon^{p_2+2}), && \forall U_0 \in (H^1)^d, 0 \le t < T_1, \\ &|U(t)|_\varepsilon \le \delta_f, && \forall |U_0|_\varepsilon < \delta, T < t < T_1. \end{aligned} \tag{4.25}$$

By (4.25) $T_1 = \infty$. Indeed, by Lemma 4.1 $\hat{F}_\varepsilon$ maps bounded sets of $(H^1)^d$ into bounded sets of $(L^2)^d$, so by Theorem 3.3.4 [8], either $T_1 = \infty$, or there is a sequence $t_n \uparrow T_1$, such that $|U(t_n)|_{H^1} \to \infty$, which contradicts (4.25).

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