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Profinite completion of groups and 3-manifolds I

Branched Coverings, Degenerations, and Related Topics

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Finite quotients

In this lecture π will be a finitely generated and residually finite group.

Let $Q(\pi)$ be the set of finite quotients of π.

What properties of π can be deduced from $Q(\pi)$?

For example if all finite quotient of π are abelian, then π is abelian.

Finite quotients of π corresponds to finite index normal subgroups of π

So properties related to finite quotients of π are encoded in the profinite completion of π.

Profinite completion

Let $\mathcal{N}(\pi)$ be the collection of all finite index normal subgroups Γ of π.

$\mathcal{N}(\pi)$ is a directed set for the following pre-order : $\Gamma' \geq \Gamma$ if $\Gamma' \subset \Gamma$.

If $\Gamma' \geq \Gamma$ there is an induced epimorphism $h_{\Gamma',\Gamma} : \pi/\Gamma' \to \pi/\Gamma$.

So to a group π one can associate the inverse system :

$$\{\pi/\Gamma, h_{\Gamma',\Gamma}\}_\Gamma \text{ with } \Gamma \in \mathcal{N}(\pi)$$

The profinite completion of π is defined as the inverse limit of this system :

$$\widehat{\pi} = \varprojlim \pi/\Gamma$$

Profinite completion

Equip each finite quotient $\pi/\Gamma, \Gamma \in \mathcal{N}(\pi)$ with the discrete topology.

The set $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ is compact .

Let $i_\pi : \pi \to \prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$ given by $\{g \in \pi \to \{g\Gamma\}_{\Gamma \in \mathcal{N}(\pi)}\}$.

Then $\widehat{\pi}$ can be identified with the closure $\overline{i_\pi(\pi)}$ in $\prod_{\Gamma \in \mathcal{N}(\pi)} \{\pi/\Gamma\}$.

$i_\pi : \pi \to \widehat{\pi}$ is injective since π is residually finite.

Profinite completion

$\widehat{\pi}$ is a compact topological group.

A subgroup $U < \widehat{\pi}$ is open if and only if it is closed and of finite index.

A subgroup $H < \widehat{\pi}$ is closed if and only if it is the intersection of all open subgroups of $\widehat{\pi}$ containing it.

Thm (N. Nikolov and D. Segal (2007))

Let π be a finitely generated group. Then every finite index subgroup of $\widehat{\pi}$ is open. In particular $\widehat{\widehat{\pi}} = \widehat{\pi}$.

Corollary

Let π be a finitely generated and residually finite group, then :

(i) A finite index subgroup $\Gamma \subset \pi \to \overline{\Gamma} \subset \widehat{\pi}$, $[\pi : \Gamma] = [\widehat{\pi} : \overline{\Gamma}]$ and $\overline{\Gamma} \cong \widehat{\Gamma}$.

(ii) Conversely an open subgroup $H \subset \widehat{\pi} \to H \cap \pi \in \pi$.

(iii) $\Gamma \trianglelefteq \pi \Leftrightarrow \overline{\Gamma} \trianglelefteq \widehat{\pi}$, and $\pi/\Gamma \cong \widehat{\pi}/\overline{\Gamma}$.

Homomorphisms

An important consequence is :

Lemma

For any finite group G the map $i_\pi : \pi \to \widehat{\pi}$ induces a bijection $i_\pi^* : Hom(\widehat{\pi}, G) \to Hom(\pi, G)$.

A group homomorphism $\varphi : A \to B$ induces a continuous homomorphism $\widehat{\varphi} : \widehat{A} \to \widehat{B}$.

If A and B are finitely generated, any homorphism $\widehat{A} \to \widehat{B}$ is continuous.

If φ is an isomorphism, so is $\widehat{\varphi}$.

On the other hand, an isomorphism $\phi : \widehat{A} \to \widehat{B}$ is not necessarily induced by a homomorphism $\varphi : A \to B$.

There are isomorphisms $\widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}$ that are not induced by an automorphism of $\mathbb{Z}$.

Isomorphisms

Let A and B be two finitely generated groups and $f : \widehat{A} \to \widehat{B}$ be an isomorphism.

For any finite group G the isomorphism $f : \widehat{A} \to \widehat{B}$ induces a bijection :

$$i_A^* \circ f^* \circ {i_B^*}^{-1} : Hom(B, G) \xrightarrow{{i_B^*}^{-1}} Hom(\widehat{B}, G) \xrightarrow{f^*} Hom(\widehat{A}, G) \xrightarrow{i_A^*} Hom(A, G).$$

Given $\beta \in Hom(B, G)$ denote by $\beta \circ f$ the resulting homomorphism in $Hom(A, G)$.

Groups A and B with isomorphic profinite completions have the same set of finite quotients : $Q(A) = Q(B)$.

The converse also holds :

Lemma

Two finitely generated groups A and B have isomorphic profinite completions if and only if they have the same set of finite quotients.

Profinite rigidity

According to Grunwald and Zaleskii let define the genus of π as :

Definition

$\mathcal{G}(\pi) = \{$*finitely generated, residually finite groups* Γ *such that* $\widehat{\Gamma} \cong \widehat{\pi}\}$, *modulo isomorphisms.*
A residually finite and finitely generated group π *is profinitely rigid if* $\mathcal{G}(\pi) = \{\pi\}$.

Question

Which groups are profinitely rigid ? Can $\mathcal{G}(\pi)$ *be infinite ?*

Surprinsingly, the following question is still open :

Question

Is a finitely generated free group profinitely rigid ?

Profinite properties

One may ask a weaker question :

Question

What group theoretic properties are shared by groups in $\mathcal{G}(\pi)$?

Such properties are called *profinite properties* of a group. For example, being abelian is a profinite property.

The next lemma says that the abelianizations are the same.

Lemma

$\widehat{\Gamma} \cong \widehat{\pi} \Rightarrow \Gamma^{ab} \cong \pi^{ab}$

Corollary

If π is abelian, then $\mathcal{G}(\pi) = \{\pi\}$

Examples

In general $\mathcal{G}(\pi) \neq \{\pi\}$

Thm (Baumslag 1974) ; Hirshon (1977))

Let Let Γ and π two finitely generated groups. If $\Gamma \times \mathbb{Z} \cong \pi \times \mathbb{Z}$ then $\widehat{\Gamma} \cong \widehat{\pi}$.

Given a group A and a class $\psi \in \mathrm{Aut}(A)$, one can build the semidirect product $A_\psi := A \rtimes_\psi \mathbb{Z}$.

It corresponds to the split exact sequence

$$1 \to A \to A_\psi \to \mathbb{Z} \to 1,$$

where the action of $\mathbb{Z}$ on A is given by ψ.

The isomorphism type of A_ψ depends only on the class of ψ in $\mathrm{Out}(A)$.

As a consequence one gets examples of finitely generated and residually finite groups which are not profinitely rigid :

Examples

Corollary

Let A be a finitely presented and residually finite group and $\psi \in \mathrm{Aut}(A)$ such that ψ^n is an inner automorphism for some $n \in \mathbb{Z}$. Then for any $k \in \mathbb{Z}$ relatively prime to n, $\widehat{A_{\psi^k}} \cong \widehat{A_\psi}$.

Example

Let $\pi_1 = \mathbb{Z}/25\mathbb{Z} \rtimes_\psi \mathbb{Z}$ and $\pi_2 = \mathbb{Z}/25\mathbb{Z} \rtimes_{\psi^2} \mathbb{Z}$, $\psi \in \mathrm{Aut}(\mathbb{Z}/25\mathbb{Z})$ be given by $\psi(x) = x^6$ for a generator $x \in \mathbb{Z}/25\mathbb{Z}$. Then $\widehat{\pi_1} \cong \widehat{\pi_2}$. In this example ψ is of order 5 in $\mathrm{Out}(\mathbb{Z}/25\mathbb{Z})$.

Since A is residually finite and finitely generated, the profinite completion $\widehat{A_\psi}$ can be computed from $\widehat{A}$ and $\widehat{\mathbb{Z}}$.

Examples

The system of characteristic finite index subgroups $C(n) := \cap_{[A:\Gamma]\leq n}\Gamma$ is cofinal in A.

For each $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that ψ^m induces the identity on the characteristic quotient $A/C(n)$.

It follows that $C(n)_{\psi^m} := C(n) \rtimes_{\psi^m} \mathbb{Z}$ is a cofinal system of normal finite index subgroups of A_ψ, since $A \cap C(n)_{\psi^m} = C(n)$.

In particular A_ψ is residually finite and its profinite topology induces that of A, so the closure $\overline{A} \subset \widehat{A_\psi}$ can be identified with $\widehat{A}$.

By using the automorphisms induced by the elements of $\mathrm{Aut}(A)$ on the finite quotients $A/C(n)$ and the equality $\widehat{A} = \varprojlim A/C(n)$, one can define an homomorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(\widehat{A})$.

Since $\mathrm{Aut}(A)$ is itself residually finite, the above homomorphism extends to a homomorphism $\widehat{\mathrm{Aut}(A)} \to \mathrm{Aut}(\widehat{A})$.

Examples

Therefore any homomorphism $\psi : \mathbb{Z} \to \text{Aut}(A)$ extends to a homomorphism $\hat{\psi}: \widehat{\mathbb{Z}} \to \widehat{\text{Aut}(A)} \to \text{Aut}(\widehat{A})$.

These are key observations for the proof of the following results :

Proposition (Nikolov-Segal 2007)

Let A be a finitely generated and residually finite group and $\psi \in \text{Aut}(A)$, then :

1. $\widehat{A_\psi} = \widehat{A \rtimes_\psi \mathbb{Z}} = \widehat{A} \rtimes_{\hat{\psi}} \widehat{\mathbb{Z}}$.
2. $\widehat{A_\psi} = \widehat{A} \times \widehat{\mathbb{Z}}$ *if and only if* ψ *induces an inner automorphisms on the finite characteristic quotients of* A

Nikolov and Segal have given an example of a finitely generated and residually finite group A with an automorphism $\psi \in \text{Aut}(A)$ such that no positive power of ψ is an inner automorphism, but $\widehat{A_\psi} = \widehat{A} \times \widehat{\mathbb{Z}}$.

3-manifold groups

In these lectures M will be a compact orientable aspherical 3-manifold with empty or toroidal boundary. For example the exterior $E(K)$ of a knot $k \subset S^3$.

By Perelman's Geometrization Theorem $\pi_1(M)$ is residually finite.

Definition

An orientable compact 3-manifold M is called profinitely rigid if $\widehat{\pi_1(M)}$ distinguishes $\pi_1(M)$ from all other 3-manifold groups.

There are closed 3-manifolds which are not profinitely rigid.

The examples known at the moment are **Sol manifolds** (P. Stebe, L. Funar), or **Surface bundle with periodic monodromy, i.e Seifert fibered manifolds** (J. Hempel).

Examples : Seifert fibered

We describe now the Seifert fibered examples given by J. Hempel.

Let F be a closed orientable surface, $h \in Homeo^{+}(F)$ and $M = F \rtimes_h S^1$ be the surface bundle over S^1 with monodromy h.

Let $h_\star \in \mathrm{Aut}(\pi_1(F))$ be the automorphism induced by h, then $\pi_1(F)_{h_\star} = \pi_1(F) \rtimes_{h_\star} \mathbb{Z} \cong \pi_1(M)$.

Proposition (Hempel 2014)

If M and N are surface bundles with periodic monodromies h and h^k, for k coprime to the order of h, then $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$.

Seifert fibered rigidity

Thm (G. Wilkes (2015))

Let M be a closed orientable irreducible Seifert fibre space. Let N be a compact orientable 3-manifold with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. Then either :

- *M is profinitely rigid, i.e. $\pi_1(N) \cong \pi(M)$, or*
- *M and N are Hempel examples.*

Corollary

Let F be a closed orientable surface. A homeomorphism h of F is homotopic to the identity if and only if it induces an inner automorphisms on every finite characteristic quotient of $\pi_1(F)$.

Does the action induced by h on all the finite characteristic quotients of $\pi_1(F)$ determine $h_\star \in Out(\pi_1(F)$ when h is not periodic ?

The next examples of torus bundles with Anosov monodromies show that it is not true.