Galois embeddings of elliptic curves and abelian surfaces

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Notation

k : ground field, $\bar{k} = k$ and $\mathrm{ch}(k) = 0$
V : nonsingular proj. variety, $\dim V = n$
D : very ample divisor
$f = f_D : V \longrightarrow \mathbb{P}^N$: embedding by $|D|$
where $N + 1 = \dim H^0(V, \mathcal{O}(D))$
W : linear subvariety of $\mathbb{P}^N$, $\dim W = N - n - 1$, $W \cap f(V) = \emptyset$
$\pi_W : \mathbb{P}^N \dashrightarrow W_0$: projection with the center W
(where W_0 linear subvariety, $\dim W_0 = n$ and $W \cap W_0 = \emptyset$)
$\pi = \pi_W \cdot f : V \longrightarrow W_0 \cong \mathbb{P}^n$
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Galois embedding

$\pi^* : K_0 \hookrightarrow K$: finite extension, $\deg = d = \deg f(V) = D^n$

The structure of this extension does not depend on W_0, but on W.

K_W : Galois closure of K/K_0

$G_W := \mathrm{Gal}(K_W/K_0)$

Remark

G_W is isomorphic to the monodromy group of $\pi : V \longrightarrow W_0$.

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Galois embedding

Definition

The V is said to have a Galois embedding if there exists a very ample divisor D s.t. the embedding by $|D|$ has a Galois subspace. In particular, if W is a point or line, we call it a Galois point or Galois line respectively.

In this case we say that (V, D) defines a Galois embedding.

Remark

Similarly we can define the Galois embedding in the case where $W \cap f(V) \neq \emptyset$.
We do not treat this case in this talk.

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Plane cubic

Example

E : smooth cubic in $\mathbb{P}^2$.
If there exists a Galois point,
then E is projectively equivalent to the curve defined by
$Y^2Z = 4X^3 + Z^3$
and it has just three Galois points
$(X : Y : Z) = (1 : 0 : 0), (0 : -\sqrt{-3} : 1)$ and
$(0 : \sqrt{-3} : 1)$. Then we have three projections
$\pi : \mathbb{P}^2 \cdots \to \mathbb{P}^1$
given by $\pi(X : Y : Z) = (Y : Z)$, $(X : Y + \sqrt{-3}Z)$ and
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Example

For any elliptic curve E there exists a Galois embedding in $\mathbb{P}^3$ whose Galois group is isomorphic to V_4.

Later we will see this in detail.

Example

The elliptic curve E with $J(E) = 1728$ has an embedding $C \subset \mathbb{P}^3$

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Then C has four Z_4-lines and three V_4-lines, the defining equations are given as follows :

(I) Z_4-liens :

① $\ell_1 : X = Y = 0$
② $\ell_2 : Z = X + 4Y = 0$
③ $\ell_3 : W = X - 4Y + 4iZ = 0$, where $i = \sqrt{-1}$
④ $\ell_4 : W = X - 4Y - 4iZ = 0$

(II) V_4-lines :

⑤ $\ell_5 : X - 4Y = Z = 0$
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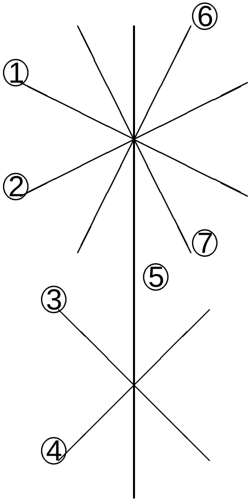

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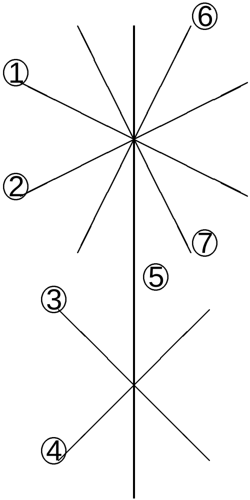

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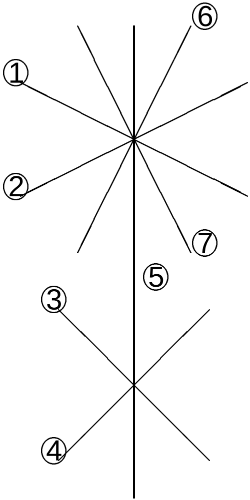
Figure



① to ④ : Z_4-lines, ⑤, ⑥ and ⑦ : V_4-lines

Remark

No divisor of degree five on elliptic curve has Galois embedding.

Problems

Problem

(1) *Find the structure of* G_W.

(2) *Find the subset* $\mathcal{S}$ *of* $Pic(V)$ *such that it consists of* D *which gives the Galois embedding.*

(3) *Find the arrangement of Galois subspaces for* $f(V)$.

(4) *For an embedding* (V, D) *find the structure of Galois group* G_W *for each* $W \in \mathrm{Grass}(N - n - 1, N)$.

(5) *How is the set* $\{ W \in \mathrm{Grass}(N - n - 1, N) \mid G_W \cong S_d \}$ *? In particular, is it true that the codimension of the complement of the set is at least two ?*

(6) *Suppose that* $\dim Lin(f(V)) = 0$, W *and* W' *are close and* $W \neq W'$. *Then is it true that* K_W *is not isomorphic to* K'_W *?*

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The results change greatly whether

(A) $ch(k) = 0$ or > 0,

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Basic results

Hereafter we assume W is a Galois subspace.

Proposition

There exists an injective representation $\alpha : G_W \hookrightarrow Aut(V)$.

Corollary

If $Aut(V)$ *is trivial, then* V *has no Galois embedding.*

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We have another injective representation $\beta : G_W \hookrightarrow PGL(N, k)$.

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We have $W_0 \cong V/G_W$
The projection $\pi : V \longrightarrow W_0$ *turns out a finite morphism.*
In particular the fixed loci of G_W *consists of divisors.*

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(V, D) *defines a Galois embedding iff*

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abelian variety

Let us apply the above method to abelian varieties.

$k = \mathbb{C}$: field of complex numbers

A : abelian variety, $\dim A = n$

G : subgroup of Aut(A)

$\sigma \in G$ has the analytic representation $\tilde{\sigma} z = M(\sigma) z + t(\sigma)$

where $M(\sigma) \in GL(n, \mathbb{C})$, $z \in \mathbb{C}^n$, $t(\sigma) \in \mathbb{C}^n$

$G_0 = \{ \sigma \in G \mid M(\sigma) = 1_n \}$,

$H = \{ M(\sigma) \mid \sigma \in G \}$

We have the following exact sequence of groups:

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow H \longrightarrow 1.$$

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Basic property 1

Assume A has the Galois embedding and let G be the Galois group.
$B = A/G_0$ is abelian variety.
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We determin the structures of G and H in the cases where $d = 1$ and 2 respectively.

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Suppose (A, D) defines Galois embedding.
Let R be the ramification divisor for $\pi : A \longrightarrow W_0$.
Then, each component of R is a translation of an abelian survariety of dimension $n-1$.
$R \sim (n+1)D$
R is very ample and $R^n = (n+1)^n|G|$.

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Corollary

Simple abelian variety A does not have Galois embedding if $\dim A \geq 2$.

elliptic curve

Let us apply the above method to elliptic curves.

$A = E$: elliptic curve

Lemma

A finite subgroup G of $Aut(E)$ can be a Galois group of some Galois embedding of E iff $|G| \geq 3$ and $|G_0| \neq 1$.

So the question is to find all finite subgroups of $Aut(E)$.
As a direct consequence the following assertion holds:

Corollary

For any smooth elliptic curve E there exists a Galois embedding whose Galois group is isomorphic to D_n.

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Bidihedral group

Definition

A finite group G is called a bidihedral group if it is generated by the elements a, b and c s.t.
(1) $a^2 = b^m = c^n = id,\ aba = b^{-1},\ aca = c^{-1}, bc = cb$
(2) $n \geq m \geq 2$ and $n \geq 3$

We denote this group by BD_{mn} or BD

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Exceptional elliptic group

Definition

A finite non-abelian group G of order m^2kl is called an exceptional elliptic group
if it satisfies the following conditions (1), (2) and (3).

(1) $l = 3, 4$ or 6

(2) G is the semi-direct product $H \rtimes K$ with some action of K onto H,
where K is a cyclic group of order l and H is the normal abelian subgroup of G of order m^2k with one or two generators such that the orders of them are m and mk respectively.

(3) In case H has one generator we regard $m = 1$.

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A finite non-abelian group G of order m^2kl is called an exceptional elliptic group if it satisfies the following conditions (1), (2) and (3).

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$k = 1$ or $k = q_1 \cdots q_s$, where q_i are distinct prime numbers satisfying the following condition (3.1) or (3.2).

(3.1) If $l = 3$ or 6, then $q_i = 3$ or $q_i \equiv 1 \pmod 3$, where $i = 1, \ldots, s$.

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Moreover, the cases (1.1), (1.2), (2.1) *and* (2.1) *appear in the cases where* $|G_0| = 1,\ |G_0| > 1,\ |G_0| = 2$ *and* $|G_0| > 2$ *respectively.*

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Make examples

Remark

By projecting an embedded elliptic curve with Galois subspace into the plane, we get a singular elliptic curve with Galois point.

Let us make examples of plane elliptic curve with a Galois point. Let G be the group in the above theorem and suppose $\mathbb{C}(x, y)^G = \mathbb{C}(s)$.

Then, taking an affine coordinate s, we have a morphism $p : E \longrightarrow E/G \cong \mathbb{P}^1$.

Let D be the polar divisor of s on E.

Next, find an element $t \in \mathbb{C}(x, y)$ *satisfying that* $\mathrm{div}(t) + D \geq 0$ *and* $\mathbb{C}(x, y) = \mathbb{C}(s, t)$.

Then, the curve C defined by s and t has the Galois point at ∞ *with the Galois group G.*

Of course C is birational to E.

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Example

Let $E : y^2 = x(x-1)(x-b)$ be an elliptic curve, where $b \neq 0, 1$.
Take the automorphisms σ and τ of $\mathbb{C}(x, y)$ such that the complex representations are $\tilde{\sigma}(z) = -z$ and $\tilde{\tau}(z) = z + \beta$, where $2\beta \in \mathcal{L}$ and $\beta \notin \mathcal{L}$.
The point $(b, 0) \in E$ is of order 2 and we have

$$(x, y) * (b, 0) = \left(\frac{b(x-1)}{x-b}, \frac{b(b-1)y}{(x-b)^2} \right).$$

Then the translation τ of order two can be expressed as

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Therefore we have $\mathbb{C}(x,y) = \mathbb{C}(s,t)$.

Thus we have the defining equation

$$a^4 + a^3(4b - 2s)t + abt(-4b + 4b^2 + 2s + 2bs - 4b^2s - 2s^2 + 2bs^2 - 4bt^2 + 4b^2t^2 + 2st^2 - 2bst^2) + a^2(2b + 2b^2 - 6bs - 2b^2s + s^2 + 4bs^2 - s^3 - 2bt^2 + 6b^2t^2 - 4bst^2 + s^2t^2) = b^2(-1 + 2b - b^2 + 2s - 4bs + 2b^2s - s^2 + 2bs^2 - b^2s^2 - 2t^2 + 4bt^2 - 2b^2t^2 + 2st^2 - 4bst^2 + 2b^2st^2 - t^4 + 2bt^4 - b^2t^4)$$

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Very ampleness

Lemma

Now, return to the case of abelian surface.
we apply the above method to abelian surfaces.
Let A be an abelian surface. Assume that G is a finite automorphism group of A
satisfying that A/G is isomorphic to $\mathbb{P}^2$
and let $\pi : A \longrightarrow \mathbb{P}^2$ be the quotient morphism.
If $\deg \pi \geq 10$, then $\pi^*(\ell) = D$ is very ample for each line ℓ in $\mathbb{P}^2$.

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If an abelian surface A has the Galois embedding, then $H = G/G_0$ is isomorphic to one of the following:

(1) D_3

(2) D_4

(3) *the semi-direct product of groups:* $Z_2 \ltimes H'$, *where* $H' \cong D_4$ *or* $Z_m \times Z_m$ $(m = 3, 4, 6)$

To state case (3) more precisely, we put $Z_2 = \langle a \rangle$ *and* $H' = \langle b, c \rangle$. *Then the actions of* Z_2 *on* H' *are as follows:*

In the former case $H' \cong D_4$, *we have* $aba = bc^2$, $aca = c$, $c^4 = 1$, $b^2 = 1$ *and* $bcb = c^{-1}$.

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Corollary

If A has a Galois embedding, then the abelian surface $B = A/G_0$ is isomorphic to $E \times E$ for some elliptic curve E.

Example 1

Example

Let A be the abelian surface with the period matrix

$$\Omega = \begin{pmatrix} -1 & \rho^2 & -\tau & \tau\rho^2 \\ 1 & \rho & \tau & \tau\rho \end{pmatrix} = \begin{pmatrix} -1 & \rho^2 \\ 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix},$$

where $\Im\tau > 0$ and $\rho = \exp(2\pi\sqrt{-1}/6)$. Clearly we have $A \cong E \times E$ where $E = \mathbb{C}/(1, \tau)$.

Letting $z \in \mathbb{C}^2$ and $\boldsymbol{v}_i$ be the i-th column vector of Ω $(1 \leq i \leq 4)$, we define t_i to be the translation on A such that $t_i z = z + \boldsymbol{v}_i/m$, where m is an integer ≥ 2.

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$$\Omega = \begin{pmatrix} -1 & \rho^2 & -\tau & \tau\rho^2 \\ 1 & \rho & \tau & \tau\rho \end{pmatrix} = \begin{pmatrix} -1 & \rho^2 \\ 1 & \rho \end{pmatrix} \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix},$$

where $\Im\tau > 0$ and $\rho = \exp(2\pi\sqrt{-1}/6)$. Clearly we have
$A \cong E \times E$ where $E = \mathbb{C}/(1, \tau)$.
Letting $z \in \mathbb{C}^2$ and $\boldsymbol{v}_i$ be the i-th column vector of Ω $(1 \leq i \leq 4)$,
we define t_i to be the translation on A such that $t_i z = z + \boldsymbol{v}_i/m$,
where m is an integer ≥ 2.
Let a and b be the automorphism of A such that the complex representations are

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\rho & 0 \\ 0 & \rho^2 \end{pmatrix}$$

respectively. Put $G_0 = \langle t_1, \ldots, t_4 \rangle$ and $G = \langle G_0, a, b \rangle$.
Then G_0 is a normal subgroup of G and $G/G_0 \cong D_3$.
Clearly we have $|G| = 6m^4$. Looking at the fixed loci of H, we infer that A/G is smooth.

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Example 2

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Let E be the elliptic curve $\mathbb{C}/\Omega$,
where $\Omega = (1, \tau)$ is a period matrix such that $\Im\tau > 0$.
Let a and b be the automorphisms of E defined by $a(z) = -z$ and $b(z) = z + 1/m$ respectively,
where $z \in \mathbb{C}$ and m is a positive integer ≥ 2.
Let G be the subgroup of $Aut(E)$ generated by a, b. Then $G = \langle a,\ b\rangle \cong D_m$; the dihedral group of order $2m$.
Let $y^2 = 4x^3 + px + q$ be the Weierstrass normal form of E and $K = \mathbb{C}(x, y)$.
Then the fixed field of K by G is rational $\mathbb{C}(t)$, where $t \in \mathbb{C}(x)$.
Putting $D = (t)_\infty$; the divisor of poles of t, we infer readily that $\deg D = 2m$ and (E, D) defines a Galois embedding for each m.

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Continuation

Example

Let E be the elliptic curve E in the example above such that $\tau = e_m,\ m = 3,\ 4 \text{ or } 6$.

Let A be the abelian surface $E \times E$. We define automorphisms on A as follows:

let a, b and c be the homomorphisms whose complex representations are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}$$

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Clearly we have $a^2 = b^m = c^m = 1,\ bc = cb,\ ca = ab$ and $ba = ac$, and $|G| = 2m^2$.

Moreover we have $G \cong Z_2 \ltimes (Z_m \times Z_m)$.

Put $E_1 = E \times \{0\}$ and $E_2 = \{0\} \times E$, where 0 is the zero element of E,

then put $D = n(E_1 + E_2)$, clearly we have $D^2 = 2n^2$.

It is well known that D is very ample if $n \geq 3$.

We see from the criterion that (A, D) defines a Galois embedding whose Galois group is isomorphic to G.

Let us examine the case $m = 3$ in a different point of view.

Since E is defined by the Weierstrass normal form

$y^2 = 4x^3 + 1$,

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Example

The automorphisms a, b and c induce the ones of $\mathbb{C}(A)$ as follows:

(1) a^* interchanges x and x', y and y'.

(2) $b^*(x) = \rho^2 x$ and b^* fixes y, x' and y'.

(3) $c^*(x') = \rho^2 x'$ and c^* fixes x, y and y'.

Therefore, the fixed field $\mathbb{C}(A)^G$ is $\mathbb{C}(y + y', yy')$,
and we have $(y + y') + D \geq 0$ and $(yy') + D \geq 0$.
Embedding by $3(E_1 + E_2)$ is the composition of the embedding $E \times E \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$

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Example

followed by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.
Using homogeneous coordinates (X, Y, Z) [resp. (X', Y', Z')] satisfying that $x = X/Z$, $y = Y/Z$ [resp. $x' = X'/Z', y' = Y'/Z'$], we can express this embedding as

$$f(X, Y, Z, X', Y', Z') = (XX', YX', ZX', XY', \ldots, ZZ').$$

Letting $(T_0, \cdots, T_8)$ be a set of homogeneous coordinates of $\mathbb{P}^8$, we can express the Galois subspace by $T_5 + T_7 = T_4 = T_8 = 0$.

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Letting $(T_0, \cdots, T_8)$ be a set of homogeneous coordinates of $\mathbb{P}^8$,
we can express the Galois subspace by $T_5 + T_7 = T_4 = T_8 = 0$.

Continuation

Example

followed by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.
Using homogeneous coordinates (X, Y, Z) [resp. (X', Y', Z')]
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In case $f(V) \cap W \neq \emptyset$, H can be abelian, in fact, in the situation above

let W be the linear subspace defined by $T_5 = T_7 = T_8 = 0$.
Consider the projection π_W with the center W.
Then $f(A) \cap W$ consists of nine points.
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Minimal embedding

If an abelian surface is embedded into $\mathbb{P}^N$, then $N \geq 4$, and in case $N = 4$ the abelian surface has a special structure.

Reider's Theorem

Theorem

Suppose L is an ample line bundle of type $(1, d)$ with $d \geq 5$ and does not split.
Then the morphism $\varphi_L : A \longrightarrow \mathbb{P}^{d-1}$ is an embedding if and only if there is no elliptic curve E on A with $(E.L) = 2$.

Similarly let us find the least number N that the abelian surface A has the Galois embedding into $\mathbb{P}^N$.
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Suppose (A, D) defines the Galois embedding. Then the least number N is seven, i.e., A is embedded into $\mathbb{P}^7$. Moreover H is isomorphic to D_4 or $Z_2 \ltimes D_4$.

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Example 3

Example

$A = \mathbb{C}^2/\Omega$, Ω is the period matrix

$$\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \tau \end{pmatrix}, \text{ where } \Im\tau > 0.$$

$$\begin{aligned}
\widetilde{g_1}\vec{z} &= \vec{z} + \tfrac{1}{2}\begin{pmatrix} n_1 + n_3\tau \\ n_2 + n_4\tau \end{pmatrix}, \\
\widetilde{g_2}\vec{z} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\vec{z} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \\
\widetilde{g_3}\vec{z} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{z}
\end{aligned}$$

where $(n_1, n_2, n_3, n_4) = (0,0,1,1), (1,1,0,0), (1,1,1,1)$, $\begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix} \in \mathcal{L}_A$ and $\begin{pmatrix} 2\alpha_1 \\ 0 \end{pmatrix} \in \mathcal{L}_A$,

Example(continuation)

Example

Then we have ${g_1}^2 = {g_2}^2 = {g_3}^4 = id$, $g_2 g_3 g_2 = {g_3}^{-1}$ and $g_i g_1 = g_1 g_i$ $(i = 2, 3)$ on A.

Putting $G = \langle g_1, g_2, g_3 \rangle$, we have $G_1 = \langle g_1 \rangle$ and $G = G_1 \times G_2$ where $G_2 = \langle g_2, g_3 \rangle$.

Clearly $G_2 \cong D_4$.

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$A = \mathbb{C}^2/\Omega$, Ω is the period matrix

$$\begin{pmatrix} 1 & 0 & i & (1+i)/2 \\ 0 & 1 & 0 & (1+i)/2 \end{pmatrix}, \text{ where } i = \sqrt{-1}.$$

Let g_1, g_2 and g_3 be the automorphisms defined by

$$\begin{aligned} \widetilde{g_1}\vec{z} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \vec{z} + \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{12} \end{pmatrix}, \\ \widetilde{g_2}\vec{z} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{z} + \begin{pmatrix} \varepsilon_{21} \\ \varepsilon_{22} \end{pmatrix}, \\ \widetilde{g_3}\vec{z} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \vec{z}. \end{aligned}$$

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Then we have
$g_1{}^2 = g_2{}^2 = g_3{}^4 = 1$, $g_1 g_2 g_1 = g_2 g_3{}^2$, $g_1 g_3 g_1 = g_3$ and $g_2 g_3 g_2 = g_3{}^{-1}$.
Putting $G = \langle g_1, g_2, g_3 \rangle$, we see that G is isomorphic to the semidirect product $Z_2 \ltimes D_4$
and G becomes a subgroup of $Aut(A)$ and $A/G \cong \mathbb{P}^2$.

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